

COLLEGE ALGEBRA

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PREFACE

The shortening of the time given to algebra in the secondary school, together with the great extension of the elective system and the consequent placing of mathematics in competition with a score of new subjects, has made some modification of the traditional college algebra absolutely necessary.

The first important change has been to put the college algebra upon a more elementary basis. Whether we like it or not, we are bound to recognize the fact that a large number of freshmen come to us with a single year of algebra, and that year so far in the past that it is of little help to the college teacher.

The second change is one which the writer believes will in the end prove to be of great advantage, not only to algebra, but to mathematics in general. The college teacher has been obliged to exert every effort to make his work of interest to his students. To this end the subject has been made more concrete, applications to the affairs of everyday life have been emphasized, processes have been made more direct, and the road to the mathematics of the sophomore year has been shortened.

In the following pages I have tried to introduce enough elementary material to meet the needs of students who have had the minimum training of the secondary schools, and at the same time I have tried to recognize the growth in mental capacity that normally takes place between the first year of the high school and the freshman year of the college. I have tried to emphasize the immediately practical side of algebra by drawing freely upon geometry, physics, the theory of investment, and other branches of pure and applied science for illustrative examples. An amount of space greater than usual

has been devoted to the study of the functions which occur most frequently in practical work. At the same time I have endeavored to arrange and classify this material in such manner that the student may acquire some skill in manipulation — an accomplishment in which the average freshman is sadly lacking.

Geometrical interpretations have been introduced at the outset and have been emphasized wherever possible. Long experience has convinced me that there is no better method to make the student think of algebra in a concrete way.

I have not hesitated to omit a number of topics that are ordinarily included in textbooks on college algebra. The algebraic solution of special types of equations belongs to the theory of equations rather than to general algebra; partial fractions can be taken up in connection with the integral calculus.

I have followed the same policy with respect to proofs of important theorems where the proofs seemed to me to be beyond the average freshman. Many theorems of this class can be taken up to better advantage at a later period.

It gives me great pleasure to acknowledge my indebtedness to my colleagues, Professors L. W. Dowling, Arnold Dresden, and H. T. Burgess, Dr. Florence Allen, Dr. T. M. Simpson, and Dr. G. R. Clements, now of the U. S. Naval Academy, for valuable suggestions made while the book was in manuscript. I am also under obligations to Professor E. R. Maurer and to Professor L. R. Ingersoll for data for problems from their respective fields.

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CHAPTER I

INTRODUCTION

1. Positive Integers. Algebra, like arithmetic, deals with numbers. The ***positive integers***, or ***natural numbers***, arise through the operation of counting.

They are denoted by the symbols

$$1, 2, 3, 4, 5, 6, \cdots$$

of arithmetic. When considered in their usual order they form the so-called ***natural scale***.*

The numbers of the natural scale may be represented geometrically by points equally spaced on a straight line. Beginning with 1, which is placed at a convenient distance from an arbitrary starting point, the numbers are attached to the equally spaced points proceeding toward the right as in Fig. 1.



FIG. 1.

2. The Number Zero. To the arbitrary starting point the symbol 0, called ***zero***, is attached. Zero is a new number to be added to the scale. When numbers are used to denote

* The natural scale has three fundamental properties: (1) it begins with a definite first symbol; (2) it has no last symbol; (3) every symbol is followed by a definite next symbol. The natural scale does not depend upon any particular mode of geometric representation. See Fine's *College Algebra*, Chapter 1.

quantity, as 3 pounds, or 7 feet, this new number denotes the absence of quantity. As a number of the scale it is simply the symbol that immediately precedes the symbol 1.

3. The Four Fundamental Operations. Upon the positive integers the ***four fundamental operations***, *addition*, *subtraction*, *multiplication*, and *division*, may be performed. Of these operations two, namely, addition and multiplication, may be performed without introducing new numbers. Subtraction and division may not always be performed, as will soon appear.

4. Subtraction and Negative Numbers. The subtraction of 5 from 8 may be considered as the operation of finding the fifth number before 8 in the scale. Without the addition of new numbers, 5 cannot be subtracted from 3, for there is no fifth number before 3 in the natural scale. Neither could we express a temperature of 2 degrees below zero. To make subtraction of integers possible in all cases, negative integers with symbols

$$-1, -2, -3, -4, \cdots$$

are introduced. These negative integers are so arranged that -1 goes before 0, -2 before -1, -3 before -2, and so on. When the positive and the negative integers, and zero are arranged as indicated above and in § 1, they form the so-called ***complete scale***.*

The numbers of the complete scale are represented geometri-



Fig. 2.

cally by extending the line in Fig. 1, to the left, and attaching the symbols -1, -2, -3, and so on, to the points equidistant from each other, starting from the zero point. The points will

* The complete scale differs from the natural scale in that it has no first number. It does, however, possess the important property of the natural scale that every symbol is followed by a definite next symbol. See ftn., p. 1.

then appear as a set of equally spaced points extending in both directions from the zero point. See Fig. 2.

5. Inequalities. Of two given integers of the complete scale, we say that one is greater which is found farther to the right in the geometric representation. Thus 7 is greater than 3, 0 is greater than -1, -3 is greater than -5. The statement that one number is greater than another is called an ***inequality;*** it is expressed symbolically by placing one of the signs $>$ or $<$ between the two numbers, in such a way that the opening is toward the greater number. Thus, the inequalities

$$7 > 3, \quad -1 < 3, \quad -5 < -3$$

mean, respectively, 7 is greater than 3, -1 is less than 3, -5 is less than -3.

6. Division and Fractions. With nothing but the numbers of the complete scale available, division would be possible only in case the divisor is contained an exact number of times in the dividend. The difficulty is obviated by the introduction of ***fractions.*** For example, the quotient of 5 by 3 is defined as that number whose product by 3 is 5, and is written $5/3$ or $\frac{5}{3}$. Thus, we have by definition

$$\frac{5}{3} \times 3 = 5.$$

To represent $5/3$ geometrically, each of the lengths 0 1, 1 2, 2 3, ... of Fig. 1 is divided into three equal parts, so that the division points are three times as numerous as before. To the



FIG. 3.

first point to the right the symbol $1/3$ is attached, to the second $2/3$, and so on. The fifth point represents $5/3$. Similarly, points to the left represent successively $-1/3$, $-2/3$, and so on. Figure 3 shows how this representation is accomplished.

7. Reduction to a Common Denominator. Each interval in the complete scale may be divided into any given number of equal subintervals. If each interval be divided into 3 subintervals, and in the same geometric representation the divisions obtained by dividing each interval into 12 subintervals be marked, it will be geometrically evident that

$$\frac{5}{3}=\frac{20}{12}.$$

The second fraction is obtained from the first by multiplying numerator and denominator by 4, and the unit is now one twelfth instead of one third.

Similarly, the fraction $7/4$ which means 7 units, each one fourth as long as the unit of the complete scale, may be expressed as $21/12$.

Two fractions are said to be reduced to a *common denominator* when they are expressed in terms of the same unit. The common denominator must be a multiple of the denominators of both fractions. Any multiple will answer, but it is usually better to use the *least common multiple.*

Fractions having a common denominator are compared by comparing their numerators. For example, we have

$$\frac{7}{4}>\frac{5}{3}, \text{ since } \frac{21}{12}>\frac{20}{12}.$$

8. The System of Rational Numbers. The totality of all quotients that can be formed by means of the numbers of the complete scale (division by zero excepted)* constitutes the so-called ***system of rational numbers.*** In other words, the system of rational numbers consists of all positive and negative integers, all positive and negative fractions, and zero.

Any number of the rational system bears a definite relation to any second number of the system in the sense that it is either greater than or less than the second. Geometrically,

* The reason for excluding division by zero will be explained later (§ 14).

the first is represented by a point farther to the right or farther to the left than the point which represents the second.*

In the rational system, the four fundamental operations, division by zero excepted, are always possible.

9. The Introduction of Literal Symbols. Formulas. Ordinary arithmetic deals principally with numbers which are expressed in terms of the nine digits and zero. Algebra, on the other hand, deals with numbers which are expressed, for the most part, by means of letters. *Numbers represented by letters are combined by means of the same operations and under the same laws as numbers of the rational system.* In this way algebraic formulas in which ordinary numbers may be substituted at any time for the letters, are obtained. For example, a body moving with uniform velocity during time t moves through the distance d given by the formula $d = vt$.

EXERCISES

1. The formula for simple interest is

$$I = Prt,$$

where P is the principal, r the rate, and t the time expressed in years. Find the interest when $P = \$125$, $r = .06$, and $t = 2\frac{7}{12}$.

2. The distance traversed by a body moving freely from rest under a uniform acceleration, is given by the formula

$$s = \frac{1}{2}at^2,$$

where s is the distance in feet, t is the time in seconds, and a is the acceleration in feet per second per second. Find s when $t = 5$, and $a = 32.2$.

3. If a freely moving body has an initial velocity of v_0, the formula for space moved over is

$$s = \frac{1}{2}at^2 + v_0t.$$

Find s when $t = 5$, $a = 32.2$, and $v_0 = 20$.

* The rational system differs in *two important particulars* from the complete scale: (1) a given number is not followed by a definite next number; (2) between any two numbers of the rational scale there is one, and consequently, an indefinite number of other numbers.

4. The formula for uniform acceleration is

$$a = \frac{v_2 - v_1}{t_2 - t_1},$$

where v_1 and v_2 are the velocities at times t_1 and t_2. Find the acceleration when the velocities at the end of 3 seconds and 7 seconds are 96.6 ft./sec. and 225.4 ft./sec., respectively.

5. The formula for compound amount is $C = P(1 + i)^n$, where P is the principal, i the rate, and n the time in years. Find the compound amount of $500 for 3 years at 5 %.

6. The formula giving the sum to which an annual deposit of P dollars drawing interest at rate i for n years will amount, is given by

$$A = P\frac{(1 + i)^n - 1}{i}.$$

Find the value of a man's savings if at the end of each year for five years he deposits $100 in a savings bank paying 4 %.

7. According to the Newtonian law of gravitation, the formula for the force exerted upon a unit mass by a mass m at distance r, is $f = km/r^2$. What is the force exerted by the earth upon a unit mass at the surface, if the mass of the earth be taken to be 614×10^{25} grams, the distance to the center 637×10^6 cm., and

$$k = \frac{1}{1543 \times 10^4}?$$

10. The Three Fundamental Laws. The four fundamental operations are carried out in accordance with three fundamental laws, which are stated as follows:

I. ***The Commutative Law.*** *The order of the terms of a sum, or of the factors of a product, may be changed without affecting the result.* In symbols, we may write, for a sum and for a product, respectively,

(1) $$a + b = b + a;$$

(2) $$ab = ba.$$

II. ***The Associative Law.*** *The terms of a sum, or the factors of a product, may be grouped in any way we choose without changing the result.* In symbols, we may write, for a sum and for a product, respectively,

$$a+(b+c)=(a+b)+c; \tag{3}$$

$$a(bc)=(ab)c. \tag{4}$$

III. ***The Distributive Law.*** *The product of one factor by another which is the sum of two or more terms is the same as the sum of the terms obtained by taking the products of the first factor into each term of the second.* In symbols,

$$a(b+c)=ab+ac. \tag{5}$$

These laws, which are perfectly obvious for integers, and even for fractions, apply without exception to all numbers with which ordinary algebra deals.

The four fundamental operations together with the three fundamental laws constitute the ***rules of reckoning*** for algebra.

Many of the simpler problems in factoring are solved by direct application of the distributive law in the form

$$ab+ac=a(b+c).$$

For example,

$$mnx-mny+mnz=mn(x-y+z).$$

Again,

$$am-an+bm-bn=a(m-n)+b(m-n)=(a+b)(m-n).$$

In this case the law is applied twice.

The distributive law does not hold for all the operations employed in algebra. For example, $(a+b)^2$ is not a^2+b^2, but is $a^2+2ab+b^2$. One of the commonest mistakes is to apply the distributive law in extraction of roots. The expression $\sqrt{a^2+b^2}$ is not equal to $a+b$, as may be seen easily by substituting numbers for a and b.

11. The Generalized Distributive Law. Applied twice, the distributive law gives

$$(a+b)(c+d)=(a+b)c+(a+b)d=ac+bc+ad+bd.$$

In a similar manner, it is easy to show that

$$(a+b)(c+d)(e+f)=ace+bce+ade+bde+acf+bcf+adf+bdf.$$

The last product is formed in accordance with the generalized distributive law which applies to any number of parentheses and which is stated as follows.

The product of any number of parentheses is equal to the sum of all possible products that can be formed by taking one and only one factor from each parenthesis.

By means of this law products of several factors may be written down by inspection.

EXERCISES

1. Write out each of the following products.

(a) $(a+b)(x+y+z)$.

(b) $(x+a)(x+b)(x+c)$.

(c) $(mn+pq)(rs+tu)$.

(d) $(\sqrt{a}+\sqrt{b})(c+d+e)$.

(e) $(a+3)(b+c+2)(5+k)$.

(f) $(a+b)(x^2-xy+y^2)$.

2. Factor each of the following expressions, noting the applications of the distributive law.

(a) $ax-bay+az$.

(b) $ax^4+2bx^3+cx^2$.

(c) $3x^3-7x^2+8x$.

(d) $ax+bx+ay+by+az+bz$.

(e) $x^2-ax-bx+ab$.

(f) $ax+ay+az+bx+by+bz+cx+cy+cz$.

3. Columns of figures may be added by beginning either at the top or at the bottom. What law justifies this?

4. Can you add a column consisting of twenty numbers by adding the sum of the first eight numbers, the sum of the last six, and the sum of the remaining six? What laws give the authority?

5. Illustrate the associative law for multiplication by considering the volume of a rectangular parallelopiped whose edges are 5, 7, and 10.

6. Show how the distributive law may be interpreted geometrically by considering the total area of two rectangles having a common altitude c and with bases a and b.

7. Show how the distributive law is used in finding the area of a trapezoid with altitude a and parallel sides b and c.

8. How is the distributive law used in finding the area of a regular polygon?

9. Verify the results obtained in Exs. 1–2 by substituting numbers for the letters.

12. Extension of Meaning of the Fundamental Operations. In arithmetic the first definitions of addition, subtraction, multiplication, and division are given for positive integers. These definitions have no meaning either for fractional or for negative numbers. For example, the product of 4 by 3 is $4 + 4 + 4$ where the multiplicand 4 is used 3 times. But the product of 4 by 2/3 cannot be interpreted in the same way.

It becomes necessary, therefore, to make new definitions whenever new numbers are introduced. When the numbers to be combined are fractions, the new definitions are given by the following formulas.

(6) *For addition:* $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$

(7) *For subtraction:* $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$

(8) *For multiplication:* $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$

(9) *For division:* $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$

These formulas are merely the expressions in symbols of the rules given in arithmetic for combining fractions. They are assumed to hold for every kind of number that is used in ordinary algebra.

13. Operations with Negative Numbers and Zero. Let a and b be any rational numbers, either integers or fractions. The fundamental operations with negative numbers and zero are defined by the following equations.

$$(10) \qquad a + (-b) = a - b. \qquad (13) \qquad a \times (-b) = -ab.$$

$$(11) \qquad a - (-b) = a + b. \qquad (14) \qquad (-a) \times (-b) = ab.$$

$$(12) \qquad a + 0 = a - 0 = a. \qquad (15) \qquad a \times 0 = 0.$$

$$(16) \qquad a \div (-b) = (-a) \div b = -\frac{a}{b}.$$

$$(17) \qquad (-a) \div (-b) = \frac{a}{b}. \qquad (18) \qquad 0 \div a = 0.$$

EXERCISES

1. Perform the indicated operations in each of the following exercises, and check each result by substituting numbers for letters.

(a) $\frac{1}{2} - (-\frac{1}{3})$.

(b) $\frac{3}{4} \times (-\frac{5}{7})$.

(c) $(-\frac{5}{6}) \times (-\frac{7}{11})$.

(d) $(-\frac{5}{6}) \div \frac{3}{11}$.

(e) $(\frac{1}{7} - .5)(-1.25)$.

(f) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.

(g) $\frac{1}{a+b} + \frac{1}{a-b}$.

(h) $x + y - \frac{y}{x+y}$.

(i) $\frac{a+b}{2} \times \frac{b}{a^2 - b^2}$.

(j) $\frac{a+b}{c+d} \times \frac{a-b}{c-d}$.

(k) $\frac{a^2 + 2a + 1}{3xy} \div \frac{a^2 - 1}{9xy}$.

(l) $\left(1 + \frac{m}{n}\right) \div \left(1 - \frac{m^2}{n^2}\right)$.

(m) $\left(\frac{y}{z} + \frac{z}{x} + \frac{x}{y}\right) \div \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

(n) $\frac{a^2 - b^2}{c^2 - d^2} \times \frac{c-d}{b-a}$.

(o) $\frac{24x}{16x^2 - 1} \div \frac{8x}{4x - 1}$.

(p) $\frac{27x^3}{x^2 - 9} \div \frac{9x}{3 - x}$.

(q) $\dfrac{1 + \frac{a}{b}}{1 - \frac{a^2}{b^2}}$.

(r) $\dfrac{1 - \frac{b}{a+b}}{1 + \frac{b}{a-b}}$.

(s) $\dfrac{\frac{a^2 + x^2}{x} - a}{\frac{1}{x} - \frac{1}{a}} \div \frac{a^3 + x^3}{a^2 - x^2}$.

2. Translate the formulas 6–18 of §§ 12 and 13 into ordinary language.

14. Division by Zero not Defined. According to the definition of § 6 the division of one number a by another number b is defined by the equation

$$\frac{a}{b} \times b = a. \tag{19}$$

If the quotient a/b be denoted by q, the defining equation takes the form $a = qb$.

If the divisor b is zero, the equation can have no meaning unless a is zero also. In this case q may have any value whatever. In either case we shall have to say for the present, at least, that *the quotient has no meaning when the divisor is zero.*

15. Parentheses. In algebraic reductions it is frequently necessary to operate with long and complicated expressions. The removal of parentheses inclosing such expressions is accomplished by means of the distributive law together with the first four definitions in § 13. The parentheses may be removed one at a time.

EXERCISES

1. Simplify each of the following expressions by removing all parentheses.

(*a*) $a - [-(b - c)]$.

(*b*) $-x - 3[y - 2(x + y)]$.

(*c*) $10 - 7[-8 - 2(16 + 7)]$.

(*d*) $(a + b - c)(x - y + z)$.

(*e*) $(x + 1) - 2\{(x + 2) + 3[(x + 3) - 4(x + 4)]\}$.

(*f*) $b^2 - b\{b + c[a(b - c) + b(c - a) + c(a - b)]\}$.

(*g*) $1 - \{1 - [1 - (1 - x)]\} + x$.

2. Inclose the last three terms of $x^2 - 7y + 4z - 8w$ in a parenthesis preceded by the negative sign.

3. Inclose the second, third, and fifth terms of

$$3abc - 4b^2c + 7a^2c + 13bc^2 - 8ac^2$$

in a parenthesis preceded by the negative sign.

4. Fill out the parenthesis on the right side of the equality

$$x^4 + 2a^2x^2 - b^2 \equiv x^4 - (\quad).$$

CHAPTER II

ALGEBRAIC IDENTITIES

16. Definitions. An ***algebraic identity*** is an equality between two algebraic expressions such that one member may be transformed into the other by means of the rules of reckoning (§ 10, Chap. I).

For example, the equality

$$(x-2)(x-3) \equiv x^2 - 5x + 6$$

is an identity since the left member may be changed into the form of the right member by simple multiplication.

When numbers are substituted for the letters in an algebraic expression the result is a ***numerical value*** of the expression.

The most important property of an algebraic identity is given by the following principle, the truth of which is assumed without proof.

If two expressions are identically equal, they are numerically equal for every set of values of the letters for which the expressions are both defined.

Thus, when $x = 0$ the identity given above reduces to $6 = 6$; when $x = 2$, or $x = 3$, it reduces to $0 = 0$.

A ***conditional equality,*** or a conditional equation, is an equality such that the two members are not numerically equal for every set of values of the letters for which they are both defined.

For example, the conditional equality $x^2 - 5x + 6 = 0$ is true for $x = 2$ and for $x = 3$, but not for $x = 0$, or for $x = 1$.

Identities are frequently written with the ordinary sign of equality. If, however, it is desired to emphasize the difference between the identity and the conditional equation, the sign $\equiv$, consisting of three bars, is used for the identity.

ILLUSTRATIVE EXAMPLES. The equality

$$\frac{x^2 - 5x + 6}{x^2 - 2x} = \frac{x - 3}{x}$$

is an identity, since the left member takes the same form as the right member when both numerator and denominator are divided by $x - 2$. The two members of the equality have the same numerical value for every value of x, except $x = 0$ and $x = 2$, the two values for which they are not both defined.

The equality

$$\frac{x^2 - 5x + 6}{x^2 - 2x} = 0$$

is not an identity, but a conditional equation, since the numerical values of the two numbers are not equal for any value of x except $x = 3$.

The equality

$$3x + 4y = 7$$

is satisfied by an infinite number of sets of values of x and y. It is, nevertheless, a conditional equality, since it is not satisfied by the system of values $x = 2$ and $y = 1$.

The equalities

$$(x - 2)(x - 3) = x^2 - 5x + 6,$$
$$a + b = b + a,$$
$$x(y + z) = xy + xz,$$
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

are all identities.

The distinction between the identity and the conditional equality is a fundamental one in mathematics. In the simpler cases the test may be made either by means of the definition or by the principle given on p. 12.

All changes in the forms of algebraic expressions are effected by means of identities; all statements of problems which require the finding of definite values for an unknown are expressed in terms of conditional equations.

The product of a coefficient and positive integral powers of one or more variables may be called a ***simple term.***

For example,

$$3\,x^2y^3z,\ 5\,ax^3y^2,\ \text{and}\ 4\,ax^4,$$

are simple terms. In the second and third examples a may be thought of either as one of the variables, or a part of the coefficient.

The sum of several simple terms is called a ***polynomial.*** Thus,

$$3\,x^2y^4 - 5\,x^3yz^2 + 14\,x^3y^2z^2$$

is a polynomial.

The ***degree of a term*** of a polynomial is the sum of the exponents of the variables. In the example just given, the degrees of the terms are 6, 6, and 7, respectively.

The ***degree of a polynomial*** is the degree of the term, or terms, of highest degree. The degree of a polynomial depends upon the particular variables under consideration.

Thus, the degree of the polynomial given above is 7 if it is looked upon as a polynomial in x, y, and z, while the degree in x and y is 6; in x and z is 5; in y and z is 4. Finally, the degree in x is 3, in y is 4, and in z is 2.

A ***rational fraction*** is the quotient of two polynomials. The ***degree of a rational fraction*** is defined to be the degree of the numerator diminished by the degree of the denominator.

For example, the degrees of the expressions

$$\frac{x^2+5\,x+6}{x+1},\ \frac{x^2+5\,x+6}{x^2+7\,x+4},\ \text{and}\ \frac{x-2}{x^2+7\,x+7},$$

are 1, 0, and -1, respectively.

EXERCISES

1. Tell which of the following equalities are identities and which are conditional equations.

(*a*) $\dfrac{x^2 - 7x + 12}{x - 3} = x - 4.$

(*b*) $\dfrac{1}{1 - x} = 1 + x + \dfrac{x^2}{1 - x}.$

(*c*) $\dfrac{x^2 + 12}{x - 3} = \dfrac{7x}{x - 3} + x - 4.$

(*d*) $\dfrac{x^2 - 7x + 12}{x - 3} = 0.$

(*e*) $\dfrac{x^2 - 7x + 12}{x - 3} = 2.$

(*f*) $\dfrac{x^2 + 12}{x - 3} = \dfrac{7x}{x - 3} + x - 2.$

2. Tell the degree in x of each of the following expressions.

(*a*) $5x^3 - 6x^2 + 14x - 16.$

(*b*) $a^2x^2 - 5x + 17.$

(*c*) $\dfrac{1}{x} + x - 2.$

(*d*) $\dfrac{x - 5}{x^3 - 15}.$

(*e*) $ax^3 + bx + c.$

(*f*) $x^3 - y^3.$

(*g*) $\dfrac{x^3 - 15}{x - 15}.$

(*h*) $\dfrac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n}.$

3. Give the degrees in x, in y, in z, and in the combinations of x, y and z, of the following expressions.

(*a*) $5x^3 - 3x^2y + 5y^3 + z.$

(*b*) $\dfrac{8x^3 - 7y^3 + 14z^3}{x + y + z}.$

(*c*) $2x^3 + 3y^3 + 4z^3 - 10xyz.$

(*d*) $\dfrac{3x^3 + 6xy}{z}.$

4. What is the degree in x and y of the expression

$$a_0\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + \cdots + a_n\,?$$

17. Some Important Identities. The following identities, which are used very frequently in making algebraic transformations, should be memorized.

The *square of a first degree expression* is expressed by the formula

(1) $$(x+a)^2 \equiv x^2 + 2\,ax + a^2.$$

The *product of the sum and the difference of two numbers* is expressed by the formula

(2) $$(x+y)(x-y) \equiv x^2 - y^2.$$

The *product of two first degree expressions* is expressed by the formula

(3) $$(x+a)(x+b) \equiv x^2 + (a+b)x + ab,$$

or, more generally,

(3 a) $$(mx+a)(nx+b) \equiv mnx^2 + (mb+na)x + ab.$$

The product

(4) $$(x-y)(x^2+xy+y^2) \equiv x^3 - y^3,$$

is important, as is also the product

(5) $$(x+y)(x^2-xy+y^2) \equiv x^3 + y^3.$$

These identities are easily verified by performing the multiplications indicated in the left members.

The formulas (2) and (4) are special cases of a more general identity that may be written in the form

(6) $$(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+y^{n-1}) \equiv x^n - y^n.$$

The fifth is a special case of the formula

(7) $$(x+y)(x^{n-1}-x^{n-2}y+\cdots+(-1)^{n-1}y^{n-1}) \equiv x^n + y^n$$

when n is an odd integer. The proof of (6) and (7) will be postponed to a later section. (See Chap. XIV, § 129, Exs. 25, 26, and 27.)

18. Rational Factors. The process of breaking up an expression into its factors depends essentially upon the recognition of the appropriate identity. Many of the most important problems in factoring may be solved by direct reference to the identities in § 17.

EXERCISES

1. Factor each of the following expressions.

(*a*) $25\,a^2 - 64$.
(*b*) $a^2b^4c^2 - \frac{1}{9}$.
(*c*) $x^{2n+2} - 4$.
(*d*) $x^2 + (p-q)x - pq$.
(*e*) $ay^2 - 7\,a^2y + 6\,a^3$.
(*f*) $3^{2n} + 7 \times 3^n + 6$.
(*g*) $(1+x)^3 - 8\,x^3$.
(*h*) $8\,x^6 - 729\,y^6$.
(*i*) $27\,a^3 + 1$.
(*j*) $a^6 - 1$.
(*k*) $a^{3r} - b^{3r}$.
(*l*) $64\,x^{3n}y^n - 729\,y^{n+3}z^6$.
(*m*) $1000\,a^3 - 8\,b^{12}c^{12}$.
(*n*) $9\,x^{2n}y^2 - 4\,z^{2m}$.
(*o*) $25\,a^4b^6 - 16$.
(*p*) $x^{12} + y^{12}$. [Hint. $x^{12} = (x^4)^3$ and $y^{12} = (y^4)^3$.]
(*q*) $x^4 + x^2y^2 + y^4$. [Hint. Add and subtract x^2y^2.]
(*r*) $(m+n)(m^2 - x^2) - (m - x)(m^2 - n^2)$.
(*s*) $a^3 - b^3 - 2\,a^2b + 2\,ab^2$.
(*t*) $x^3 - 8 - 6\,x^2 + 12\,x$.
(*u*) $x^3 + 3\,x^2 - 2$. [Hint. Complete the cube of $x + 1$.]

2. Express the following as the products of first degree factors.
(*a*) $bc(b-c) + ca(c-a) + ab(a-b)$.
(*b*) $a^2(b-c) + b^2(c-a) + c^2(a-b)$.
(*c*) $(a-b)^3 + (b-c)^3 + (c-a)^3$.
(*d*) $4\,a^2c^2 - (a^2 - b^2 + c^2)^2$.
(*e*) $a^3(b-c) + b^3(c-a) + c^3(a-b)$.

3. Prove that

$$1 + \frac{b^2 + c^2 - a^2}{2\,bc} \equiv \frac{(b+c+a)(b+c-a)}{2\,bc}.$$

4. Prove that

$$1 - \frac{b^2 + c^2 - a^2}{2\,bc} \equiv \frac{(a-b+c)(a+b-c)}{2\,bc}.$$

5. Prove that

$$\frac{b^2c^2}{4}\left(1 + \frac{b^2 + c^2 - a^2}{2\,bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2\,bc}\right) \equiv s(s-a)(s-b)(s-c),$$

where $s = (a+b+c)/2$.

19. Reduction of a Quadratic Expression to the Sum or Difference of two Squares. Expressions of the form

$$x^2 + px + q, \text{ and } ax^2 + bx + c,$$

are called ***quadratic expressions.*** Whether or not such an expression can be resolved into the product of rational factors may always be determined by reducing it to the sum or difference of two squares. The sum $x^2 + px$ differs from the square of $x + p/2$ by the term $p^2/4$. If, therefore, $p^2/4$ be added to, and subtracted from, the first quadratic expression, we have

$$(8)\qquad x^2 + px + q \equiv x^2 + px + \frac{p^2}{4} - \left(\frac{p^2}{4} - q\right)$$
$$\equiv \left(x + \frac{p}{2}\right)^2 - \left(\frac{p^2}{4} - q\right).$$

If $p^2/4 > q$, $p^2/4 - q$ is a positive number; hence it can be written in the form

$$\frac{p^2}{4} - q = \left(\sqrt{\frac{p^2}{4} - q}\right)^2.$$

It follows that

$$(9)\qquad x^2 + px + q \equiv \left(x + \frac{p}{2}\right)^2 - \left(\sqrt{\frac{p^2}{4} - q}\right)^2.$$

If, on the other hand, $p^2/4 < q$, $q - p^2/4$ is positive, and the identity may be written in the form

$$(10)\qquad x^2 + px + q \equiv \left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right),$$

or

$$(11)\qquad x^2 + px + q \equiv \left(x + \frac{p}{2}\right)^2 + \left(\sqrt{q - \frac{p^2}{4}}\right)^2.$$

To sum up the results, we may state the following theorem.

The quadratic expression $x^2 + px + q$ may be expressed as the sum or the difference of the squares of two real quantities according as $p^2/4 < q$ or $p^2/4 > q$.

It follows directly that a quadratic expression can be resolved into *real* first degree factors only when $p^2/4 > q$.

EXERCISES

1. Reduce each of the following expressions to the sum or difference of two squares, and factor if possible.

(*a*) $x^2 - 6x + 8$. (*c*) $x^2 + 6x + 7$. (*e*) $x^2 - 7x + 23$.
(*b*) $x^2 - 5x + 6$. (*d*) $x^2 + 6x + 11$. (*f*) $x^2 + ax + b$.

2. Factor the expression $3x^2 - 5x + 6$.

[HINT. $3x^2 - 5x + 6 = 3(x^2 - \frac{5}{3}x + 2)$.]

20. Highest Common Factor. A common factor of two numbers a and b is a factor that is contained in both numbers. The largest common factor contained in both numbers is called the ***highest common factor*** (*H. C. F.*) of the two numbers. Thus, 2, 3, and 6 are common factors of 18 and 24, while 6 is the H. C. F. Two numbers having no common factor except unity are said to be ***prime*** to each other.

Similarly, two polynomials that have common factors, have a common factor of highest degree. If we ignore mere numerical factors, two polynomials have but one common factor of highest degree. This common factor is called the ***highest common factor.*** If the degree of the H. C. F. is zero, *i.e.* if the H. C. F. is a mere numerical factor, the two polynomials are said to be ***prime*** to each other. The polynomials $6x^3 - 30x^2 + 36x$ and $15x^3 - 105x^2 + 150x$ have as H. C. F. $x(x-2)$, while the polynomials $6x^3 - 30x^2 + 36x$ and $2x^2 - 12x + 10$ are prime to each other.

The highest common factor of two polynomials is the product of all their prime factors (numerical factors excluded), each factor being raised to the lowest power in which it occurs in either polynomial.

It is possible, therefore, to find by inspection the H. C. F. of

two numbers or of two polynomials as soon as their prime factors are known. The foregoing considerations apply without change to three or more numbers, or to three or more polynomials.

EXERCISES

1. Find the H. C. F. of the following.

(*a*) $(x+y)^2(x-y)$, $(x-y)^2(x+y)$.

(*b*) x^3-1, x^2-7x+6, $(x-1)^3$.

(*c*) $(a-b)^4$, $a^4-2a^2b^2+b^4$, $a^3-a^2b-ab^2+b^3$.

2. Reduce the following fractions to lowest terms.

(*a*) $\dfrac{2x^2+5x+3}{5x^2+12x+7}$. (*c*) $\dfrac{1+5x+6x^2}{1+6x+8x^2}$. (*e*) $\dfrac{a^4+a^2b^2+b^4}{a^6-b^6}$.

(*b*) $\dfrac{x^2-(y-z)^2}{(x+y)^2-z^2}$. (*d*) $\dfrac{x^3+3x^2+3x+2}{x^3-2x^2-2x-3}$. (*f*) $\dfrac{a-b}{a^{1/3}-b^{1/3}}$.

21. Lowest Common Multiple. A number which is divisible by each of several given numbers is called a ***common multiple*** of those numbers. If it is the least of such numbers, it is called the ***lowest common multiple*** (***L. C. M.***).

Similarly, the *L. C. M. of two or more polynomials* is that polynomial of lowest degree among all the polynomials which are divisible by the given polynomials.

The L. C. M. of several polynomials is the product of all the different prime factors (numerical factors included) of all the polynomials, each factor raised to the highest power in which it occurs in any one of the polynomials. We are thus enabled to find the L. C. M. by inspection when the numbers or the polynomials are broken up into their prime factors.

EXERCISES

1. Find the L. C. M. of the expressions

$$x-1,\ (x-2)(x-1)^2,\ (x-1)^2(x-2)^2.$$

SOLUTION. The prime factors occurring are $x-1$ and $x-2$, and the highest power to which each occurs is the second, consequently the L. C. M. is $(x-1)^2(x-2)^2$.

2. In each of the following cases, find the L. C. M. of the given expressions.

(*a*) $4x-1$, $16x^2-1$, $16x^2-8x+1$.
(*b*) $(x+y)(x^3-y^3)$, $(x-y)(x^3+y^3)$.
(*c*) x^3+x^2+x, x^5-x^3, x^6-x^3.
(*d*) x^3+y^3, x^3-y^3, $x^4+x^2y^2+y^4$.

3. The earliest application of the lowest common multiple is found in the reduction of fractions to equivalent fractions with the *lowest common denominator.*

Perform the operations indicated on the following fractions.

(*a*) $\dfrac{1}{3a-5b}+\dfrac{1}{3a+5b}-\dfrac{6a}{9a^2-25b^2}$.

(*b*) $\dfrac{a}{(b-c)(c-a)}+\dfrac{b}{(c-a)(a-b)}+\dfrac{c}{(a-b)(b-c)}$.

4. Prove that the L. C. M. of two numbers is equal to their product divided by their H. C. F.

MISCELLANEOUS EXERCISES

1. Add the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, first by using the least common denominator, and then by using the common denominator 144. Compare the form of the results.

2. Add the fractions

$$\frac{b}{ax+ab},\ \frac{a}{x^2-b^2},\ \frac{c}{bx-ab}.$$

3. Add $\dfrac{m}{y(x-y)}$, $\dfrac{y}{m(y-x)}$, and $\dfrac{1+m}{my}$.

4. Reduce $\dfrac{x^2-6x+8}{x^2+2x+1}\times\dfrac{x+1}{x-4}$ to its lowest terms.

5. Reduce $\dfrac{x^2-6x+8}{x^2-9}\times\dfrac{x^2-5x+6}{x^2-9x+20}\times\dfrac{x^2-7x+12}{(x-2)^2}$ to its lowest terms.

6. If M and N are two polynomials having a common factor, and the degree of N is less than that of M, prove that the remainder after dividing M by N is divisible by the common factor.

[HINT. $M/N=Q+R/N$ gives $M=NQ+R$.]

CHAPTER III

POWERS AND ROOTS

22. Integral Powers. The expression a^n is called the ***nth power of a.*** When n is a positive integer, a^n is defined to mean the product of n factors each equal to a. Thus, a^3 is a short way of writing the product $a \cdot a \cdot a$. The number a is called a ***base*** and the number n an ***exponent.*** For the present, powers whose exponents are negative numbers, fractions, or zero, have no meaning.

For powers with positive integral exponents, five theorems, called the ***fundamental laws for exponents,*** may be demonstrated.

I. $$a^m a^n = a^{m+n}.$$

This law is known as the ***index law.***

II. $$a^m \div a^n = a^{m-n}, \ (m > n).$$

III. $$(a^m)^n = a^{mn}.$$

IV. $$(ab)^n = a^n b^n.$$

V. $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

EXERCISES

1. **Demonstrate each of the five fundamental laws.**

2. **Translate each of the above formulas into words.**

3. **Prove that** $((a^m)^n)^r = a^{mnr}$.

4. **Prove that** $(abc)^n = a^n b^n c^n$.

5. Perform the indicated operations in each of the following exercises, and put results in their simplest forms.

(*a*) $(-2x)^5$.

(*b*) $-(6x^2y^3z)^3$.

(*c*) $\left(\dfrac{-a^2}{3b^3}\right)^4$.

(*d*) $\dfrac{(x^2)^4-(y^2)^4}{(x^2)^2+(y^2)^2}$.

(*e*) $(2^n3^n)^3$.

(*f*) $\dfrac{2}{3}\left(\dfrac{3}{4}a^2b^3c^2+\dfrac{3}{5}a^3b^2c^3\right)\div 4a^2b^2c$.

(*g*) $x^{5-n}y^{7-n}\times x^{2+n}y^{4+n}$.

(*h*) $(x+y-z)^3\times(x+y-z)^2$.

(*i*) $\dfrac{(x-y)^{10}}{(x+y)^6}\times\dfrac{(x+y)^5}{(x-y)^9}$.

(*j*) $(a^2-3a^3b)\times(a^3+2ab^2)$.

23. Zero Exponents. To find a meaning for a^0, we assume that the index law holds for all exponents. Consequently,

$$a^na^0=a^{n+0}=a^n.$$

But the only multiplier that leaves a number unchanged is 1. Therefore

$$a^0=1 \qquad (a\neq 0). \tag{1}$$

24. Negative Exponents. Again, assuming the index law to hold whatever a^n may mean,

$$a^na^{-n}=a^0=1.$$

Consequently, division of both sides by a^n gives

$$a^{-n}=\frac{1}{a^n}. \tag{2}$$

Two numbers whose product is unity are called ***reciprocals.*** According to this definition a^{-n} is the reciprocal of a^n and *vice versa.* On account of (2) a number which is a factor of one term of a fraction may be removed to the other side of the line if the sign of its exponent be changed. Thus,

$$\frac{5^{-2}\cdot 6}{17}=5^{-2}\cdot\frac{6}{17}=\frac{1}{5^2}\cdot\frac{6}{17}=\frac{6}{5^2\cdot 17}.$$

Similarly,

$$\frac{a^2(x^3+y^3)}{b^3}=\frac{x^3+y^3}{a^{-2}b^3}.$$

25. Fractional Exponents. The meaning of a power with a fractional exponent is determined by means similar to those employed in finding meanings for powers with zero and negative exponents. Assuming fractional powers to be defined in such manner that the index law holds, we would have, for example,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2.$$

From this it is seen that $a^{2/3}$ is *one of three equal factors* of a^2. But one of three equal factors of a number is the cube root of the number. Therefore,

$$a^{2/3} = \sqrt[3]{a^2}.$$

For the general case,

$$a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdots \text{to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\cdots \text{ to } n \text{ terms}} = a^m;$$

that is,

$$(3) \qquad a^{m/n} = \sqrt[n]{a^m}.$$

Meanings * have been found for powers with zero, negative, and fractional exponents by assuming that they follow a single one of the five fundamental laws for positive integral exponents, namely, the index law. If only positive bases are considered, it is possible to prove that all powers with rational exponents follow the remaining four laws. However, in order to save space, we shall assume without proof the following fundamental principle.

Powers of positive† bases with zero, negative, and fractional exponents obey the five fundamental laws for powers with positive integral exponents.

* The student should be careful to note that the meanings for zero, negative, and fractional exponents are essentially definitions since they are based upon the unproved assumption that the index law holds.

† All the laws except IV hold for negative, as well as for positive, bases. For the present it is sufficient to note that $[(-2)(-3)]^{1/2}$ is *not* equal to $(-2)^{1/2}(-3)^{1/2}$. This exceptional case will be considered in § 33.

THEOREM. *The nth root of the mth power of a positive number is numerically equal to the mth power of the nth root of the number.*

PROOF. It is necessary to prove that $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$. Since all powers of positive numbers with rational exponents obey the same laws,

$$a^{\frac{m}{n}} = a^{\frac{1}{n}\times m} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m,$$

if $a > 0$. But by definition, $a^{m/n} = \sqrt[n]{a^m}$; therefore, if $a > 0$,

$$(\sqrt[n]{a})^m = \sqrt[n]{(a)^m},$$

since both are equal to $a^{m/n}$.

In simplifying expressions containing exponents, it is practically always simpler to apply the laws first, and to insert the meanings in terms of radicals afterwards.

EXERCISES

1. Free each of the following expressions from zero and negative exponents.

(a) $\dfrac{2x^{-4}}{a^{-1}b^2}$. (b) $3^{-2}a^3y^{-4}$. (c) $5x^{-2}y^{-4}$.

(d) $\dfrac{a^2b}{c^{-1}+d^{-1}}$. (e) $\dfrac{x^{-2/3}y^{-2/3}}{a^{-2/3}-b^{-2/3}}$. (f) $\dfrac{x^{-2/3}-y^0}{a^{-2/3}b^0}$.

2. Perform the following indicated multiplications, and express each answer in terms of powers with positive exponents.

(a) $a^3b^{-2}\times a^{1/3}b^{-2/5}$. (b) $-\frac{5}{6}x^{-5/6}y^{2/3}\times 2x^{5/6}y^{1/2}$. (c) $3^{-2}x^{-6}\times 6^{-2}x^4$.

(d) $(a^{1/2}+x^{-2})(a^{1/2}-x^{-2})$. (e) $(x^{1/2}+x^{-1/2})(x^{1/2}-x^{-1/2})$.

(f) $(a^{-5}+a^{-3}+a^{-1})(a^{-2}-a^0)$.

(g) $(x^3y^{-3/4}+x^2y^{-1/2}+xy^{-1/4}+y^0)(xy^{-1/4}-y^0)$.

3. Perform the indicated divisions and express each result with positive exponents.

(a) $7x^{-3/4}y^{2/3}\div 2x^{-5}y$. (b) $xy^p\div x^{m/n}y^{p/q}$. (c) $\dfrac{7c^{-3}}{3a^3}\div\dfrac{35a^4}{6c^4}$.

(d) $(x^{1/2}-y^{1/2})\div(x^{1/4}+y^{1/4})$. (e) $(x-y)\div(x^{1/4}+y^{1/4})$.

4. Perform the operations indicated and express each result with positive exponents.

(*a*) $[(2-x)^{-2}-(2-x)^0][(2-x)^{-1}+(2-x)]$.

(*b*) $(x^{-4}-y^{-4})(x^{-1}-y^{-1})$.

(*c*) $(6x^{-3/5}-7x^{-2/5}-13x^{-1/5}+3)\div(2x^{-1/5}+3)$.

5. Find numerical values of each of the following expressions to three decimal places.

(*a*) $169^{1/2}$. (*b*) $512^{2/3}$. (*c*) $(.04)^{3/2}$.

(*d*) $(.0625)^{3/4}$. (*e*) $(.16)^{.5}-(1\frac{2465}{4096})^{.25}$. (*f*) $(.0625)^{-.25}$.

6. Prove that $(ab)^{-m}=a^{-m}b^{-m}$.

7. Find the numerical value of $2x^2-5x+16$ when $x=2+3(2^{1/2})$, and also when $x=2-3(2)^{1/2}$. Compare the results.

8. Prove that $(a^{-m})^{-n}=a^{mn}$.

9. Prove that $(abc)^{-m}=a^{-m}b^{-m}c^{-m}$.

10. Prove that $a^{r/s}\div a^{t/u}=a^{r/s-t/u}$.

[Hint. Find a common base with integral exponents by reducing the fractional exponents to a common denominator.]

11. Prove that $(a^{r/s})^t=a^{rt/s}$.

26. Radicals. An expression of the form $a\sqrt[n]{b}$, where b is not a perfect nth power, is called a ***radical.*** The number under the radical sign is called the ***base,*** or the ***radicand*** ; n is called the ***index,*** and a is called the ***coefficient,*** of the radical. For the present, it will be assumed that b is a positive rational number, and that $\sqrt[n]{b}$ has a single real positive value which, alone, will be considered. Numbers so defined belong to a very important class of numbers known as ***irrational numbers.****

* The irrational numbers constitute an entirely new class for which the fundamental operations must be defined anew. The fact that numbers like $\sqrt{2}$ and $\sqrt[3]{5}$ do not belong to the system of rational numbers was known to the ancient Greek geometers. Their proof was substantially as follows: Suppose, for example, that $\sqrt{2}=x/y$, where x and y are integers that have no common factor. Then $2y^2=x^2$, and x^2 must contain the factor 2. Since it is a square, x^2 must contain the factor 4, and therefore x must contain the factor 2. Consequently y^2 contains the factor 2, and so the factor 4. But if y^2 contains the factor 4, y contains the factor 2, and x and y have a common factor, which is contrary to the hypothesis. Therefore, since $\sqrt{2}$ cannot be the quotient of two integers, it is not a number of the rational system.

The system of rational numbers (§ 9, Chap. I) together with all the irrational numbers form the ***system of real numbers.*** Any number of the real system may be represented geometrically by a point of a straight line in which any arbitrary point is taken as the origin.

Clearly, all statements involving radicals may be translated into the language of fractional exponents, and *vice versa*, so that the theory of radicals is practically already known.

27. Reduction of Radicals to Other Forms. The following four reductions are frequently employed in working with radicals.

1. *Reduction to an equivalent radical with coefficient unity.* To perform this reduction it is only necessary to note that

$$a = \sqrt[n]{a^n} = (a^n)^{1/n}.$$

As a particular example,

$$3\sqrt{5} = (3^2)^{1/2}5^{1/2},$$

whence by IV, § 22,

$$3\sqrt{5} = (3^2)^{1/2}5^{1/2} = (3^2 \cdot 5)^{1/2} = \sqrt{3^2 \cdot 5}.$$

2. *Reduction to an equivalent radical with the same index, but with smaller radicand.*

This reduction is the reverse of the preceding one, and can be performed only when the radicand contains a power whose exponent is equal to the index of the radical. For example,

$$\sqrt{63} = \sqrt{3^2 \cdot 7} = (3^2)^{1/2}7^{1/2} = 3\sqrt{7}.$$

3. *Reduction to an equivalent radical with smaller index.*

For example,

$$\sqrt[6]{25} = (25)^{1/6} = (25^{1/2})^{1/3},$$

by III, § 22. Consequently,

$$\sqrt[6]{25} = (5)^{1/3} = \sqrt[3]{5}.$$

Query. Under what circumstances is this reduction possible?

4. *Reduction of a radical with a fractional radicand to another with an integral radicand.*

For example, by V, § 22,

$$\sqrt[3]{\frac{3}{5}} = \sqrt[3]{\frac{3 \cdot 5^2}{5^3}} = \left(\frac{3 \cdot 5^2}{5^3}\right)^{1/3} = \frac{(3 \cdot 5^2)^{1/3}}{5} = \frac{\sqrt[3]{75}}{5}.$$

This reduction is always possible.

A radical with index n is said to be in its ***simplest form*** when, (1) no factor of the radicand is a perfect nth power, (2) the radicand is an integer, (3) the index is as small as possible.

EXERCISES

1. Reduce each of the following expressions to a radical with unit coefficient.

(a) $5\sqrt{7}$. (c) $2\sqrt[n]{5}$. (e) $(x-a)\sqrt{(x+a)^2}$

(b) $3\sqrt[3]{2}$. (d) $(a+b)\sqrt{c}$. (f) $(x-a)\sqrt[3]{a^2}$.

2. Reduce each of the following expressions to a radical with its radicand as small as possible.

(a) $\sqrt{72}$. (d) $\sqrt[3]{108}$. (g) $\sqrt{ax^2-2axy+ay^2}$.

(b) $\sqrt{512}$. (e) $\sqrt{a^5b^3}$. (h) $\sqrt[n]{a^{n+2}b}$.

(c) $\sqrt{6^2+9^2}$. (f) $\sqrt[3]{a^5b^4}$. (i) $\sqrt{a^4+4a^2b^2}$.

3. Reduce each of the following expressions to a radical with a smaller index.

(a) $\sqrt[4]{25}$. (c) $\sqrt[2n]{a^n}$. (e) $\sqrt[6]{3^2+4^2}$.

(b) $\sqrt[6]{125}$. (d) $\sqrt[2n]{a^2}$. (f) $\sqrt[8]{256a^2b^8}$.

4. Reduce the following to radicals with integral radicands.

(a) $\sqrt{\frac{2}{3}}$. (b) $\sqrt[3]{\frac{5}{7}}$. (c) $\sqrt[n]{\frac{a}{b}}$. (d) $\sqrt[3]{\frac{a^2}{b^2}}$. (e) $\sqrt{\frac{1}{4}+\frac{4}{7}}$. (f) $\sqrt{\frac{a}{b^2}+\frac{c}{d^2}}$.

5. Reduce each of the following expressions to its simplest form.

(a) $\sqrt{25a^2-125b^2}$. (c) $\sqrt[3]{\frac{m^4n^{10}}{r^7s^8}}$. (e) $\sqrt[4]{1875x^{11}y^{13}}$.

(b) $\frac{1}{\sqrt{3}}$. (d) $\sqrt{1-\frac{1}{a^2}}$. (f) $\sqrt[3]{125+1000}$.

28. The Fundamental Operations upon Radicals. Two radicals of the form $a\sqrt[n]{b}$ are said to be ***similar*** if they have the same index and the same radicand. Thus, $3\sqrt{5}$ and $10\sqrt{5}$ are similar, as are $5\sqrt[3]{7}$ and $19\sqrt[3]{7}$. On the other hand, $3\sqrt{5}$ and $3\sqrt{6}$ are not similar.

Only radicals that are similar, or that can be made similar by means of the reductions of § 27, can be added or subtracted in the sense that they can be combined into a single radical. The addition or subtraction of dissimilar radicals can only be indicated by the proper sign. Hence the rule is as follows.

To add or subtract radicals, reduce all of them to their simplest forms, collect those that are similar, and connect those that are dissimilar by the appropriate sign.

In order to multiply or divide one radical by another it is only necessary to change each radical into the corresponding fractional power. Practice will suggest many short cuts. For example, $\sqrt{5} \times \sqrt{7} = 5^{1/2} \cdot 7^{1/2} = (35)^{1/2} = \sqrt{35}$. Clearly, in the general case the product $\sqrt{a} \times \sqrt{b}$ can be written in the form $\sqrt{ab}$. Again,

$$\sqrt{5} \times \sqrt[3]{7} = 5^{1/2} \cdot 7^{1/3} = 5^{3/6} \cdot 7^{2/6} = (5^3 \cdot 7^2)^{1/6} = \sqrt[6]{6125}.$$

EXERCISES

1. Perform the operations indicated in each of the following exercises.

(*a*) $\sqrt{50} + \sqrt{98} - \sqrt{128}$.

(*b*) $\sqrt{112\,ab^2x} + \sqrt{252\,a^3b^4x^3}$.

(*c*) $\sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}} - \frac{1}{2}\sqrt{3}$.

(*d*) $\sqrt[4]{\frac{324}{2304}} + \sqrt[6]{\frac{9}{121875}}$.

(*e*) $\sqrt[m]{a^{2m}b} - \sqrt[m]{bx^{2m}}$.

(*f*) $\sqrt{5} \times \sqrt{11}$.

(*g*) $\sqrt[3]{5} \times \sqrt[3]{11}$.

(*h*) $\sqrt{5} \times \sqrt[3]{11}$.

(*i*) $\sqrt{5} \div \sqrt{11}$.

(*j*) $\sqrt{5} \div \sqrt[3]{11}$.

(*k*) $(2 + \sqrt{3})^2$.

(*l*) $(2 + \sqrt{3})(2 - \sqrt{3})$.

(*m*) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

(*n*) $(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{a} - \sqrt[3]{b})$.

(*o*) $(a - \sqrt[3]{b})(a^2 + a\sqrt[3]{b} + \sqrt[3]{b^2})$.

(*p*) $(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})$.

(*q*) $(a + \sqrt[3]{b})(a^2 - a\sqrt[3]{b} + \sqrt[3]{b^2})$.

(*r*) $(\sqrt[3]{a} - \sqrt[3]{b})(\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2})$.

2. Frame a rule for multiplying radicals having the same index without changing to fractional exponents.

3. Prove that the *mn*th root of a number may be found by first extracting the *m*th root and then the *n*th root of the result. Show that the order of root extractions can be reversed.

4. Prove that when $a+\sqrt{b}$ is substituted for x in the quadratic expression Ax^2+Bx+C, the result may be written in the form $P+Q\sqrt{b}$, and when $a-\sqrt{b}$ is substituted the result has the form $P-Q\sqrt{b}$.

29. Radicals in the Denominator. Radicals occurring in the denominator of an expression are frequently a great hindrance to rapid numerical computation. If the radicals are parts of monomial expressions, they may be removed from the denominator by the fourth reduction of § 27. If the denominator is a binomial containing only square roots, the radicals will disappear from it when numerator and denominator are multiplied by a radical obtained from the denominator by changing the sign of one term. For example,

$$\frac{3}{2+\sqrt{3}}=\frac{3(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}=\frac{6-3\sqrt{3}}{4-3}=6-3\sqrt{3}.$$

If the denominator is of the form

$$a\sqrt{b}+c\sqrt{d}+e\sqrt{f},$$

the number of radicals may be reduced to one by using as a multiplier any one of the radical expressions

$$a\sqrt{b}+c\sqrt{d}-e\sqrt{f},$$

$$a\sqrt{b}-c\sqrt{d}+e\sqrt{f},\quad \text{or}\quad -a\sqrt{b}+c\sqrt{d}+e\sqrt{f}.$$

The two radical expressions

$$a+b\sqrt{c}\quad \text{and}\quad a-b\sqrt{c},$$

which differ only in the sign of the radical part, are called ***conjugate radicals.***

EXERCISES

1. Reduce to equivalent radicals with rational denominators.

(*a*) $\dfrac{1}{2-\sqrt{3}}$.

(*b*) $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{3}}$.

(*c*) $\dfrac{1}{2-\sqrt[3]{5}}$. [HINT. Compare with Ex. 1 (*o*), p. 29.]

(*d*) $\dfrac{5}{2+\sqrt[4]{3}}$.

(*e*) $\dfrac{3}{\sqrt{3}+\sqrt[4]{5}}$.

(*f*) $\dfrac{6}{\sqrt{3}+\sqrt{5}+\sqrt{7}}$.

(*g*) $\dfrac{3}{\sqrt{6}+\sqrt{5}-\sqrt{7}}$.

(*h*) $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$.

(*i*) $\dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$.

2. Reduce

$$\frac{x+3}{x+2\sqrt{x^2-5x+6}}$$

to the form $X_1 + X_2\sqrt{x^2-5x+6}$, where X_1 and X_2 are rational. What are X_1 and X_2?

3. Reduce the expression

$$\frac{3+5\sqrt{x^2-5x+6}}{x+2}$$

to an equivalent expression with a rational numerator.

30. Approximate Rational Values for Radicals. So far, a radical like $\sqrt{2}$ is merely a symbol for a number whose square is 2. It cannot be expressed as a rational number (ftn., § 26), and, for that reason, cannot be utilized directly in operations involving measurement. For example, a carpenter could not cut a board whose length is to be $10\sqrt{2}$ feet without a preliminary computation, and even then he would secure only a rough approximation.

The ordinary process for extracting the square root of 2 leads to a succession of numbers,

$$(10) \qquad 1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421, \cdots$$

each of which is a closer approximation to $\sqrt{2}$ than the pre-

ceding number. To see that this is true, it is only necessary to write the series of inequalities

$$\begin{aligned} 1. \quad &< \sqrt{2} < 2. \\ 1.4 \quad &< \sqrt{2} < 1.5. \\ 1.41 \quad &< \sqrt{2} < 1.42. \\ 1.414 \quad &< \sqrt{2} < 1.415. \\ 1.4142 &< \sqrt{2} < 1.4143. \\ &\cdots\cdots \end{aligned}$$

The square of every number on the left is less than 2, while the square of every number on the right is greater than 2. Moreover, the differences between the right-hand and the left-hand number in the successive inequalities are 1, .1, .01, .001, .0001, $\cdots$. It follows, therefore, that

$$\sqrt{2} - 1 < 1, \qquad \sqrt{2} - 1.4 < .1, \qquad \sqrt{2} - 1.41 < .01, \cdots.$$

In other words, each successive rational number of (10) is a better approximation to $\sqrt{2}$ than the preceding number.

A succession of numbers like (10), each of which is derived from the preceding number by a definite law, is called a ***sequence.***

The irrational numbers are brought into working relation with rational numbers (which alone can be used in measurement) by the following fundamental principle.

Corresponding to every irrational number there exists a sequence of rational numbers such that numbers of the sequence may be found which differ from the irrational number by an amount as small as we please.

31. Tables of Square and Cube Roots. In finding rational approximations for expressions containing radicals, much labor may be saved by the use of tables. By means of the table on p. 254, squares, cubes, square roots, and cube roots to the third decimal place may be found directly for all numbers

from 1 to 100, and by interpolating, the tables may be used for numbers up to 1000.

All radical expressions should be brought into the most convenient form before the tables are used. For example, to find an approximate rational value for $1/(4+\sqrt{13})$, the expression should first be reduced by the method of § 29 to the form $(4-\sqrt{13})/3$, thus avoiding the laborious division that must be carried out if the tables are used before the reduction.

EXERCISES

1. Find the value to the third decimal place of each of the following expressions.

(*a*) $\sqrt{8.15}$.

(*b*) $\sqrt{81.5}$.

(*c*) $\sqrt{815}$.

(*d*) $\sqrt{8150}$.

(*e*) $\dfrac{2}{\sqrt{54}}$.

(*f*) $\dfrac{3}{14-\sqrt{29}}$.

(*g*) $\dfrac{6}{\sqrt{17.5}-\sqrt{13.5}}$.

(*h*) $\sqrt[3]{384.2}$.

(*i*) $\dfrac{1}{\sqrt{7}\text{[illegible]}}$.

bers, $\sqrt{-5}$,

2. The time in seconds in which a body falls freely from [illegible] is gi[illegible] by the formula

$$t=\sqrt{\frac{2s}{g}},$$

where $g=32.2$. Find the time to the nearest tenth of a [illegible]nd required for a body to fall 100 feet.

3. The time in seconds of one swing of a pendulum [illegible] given approximately by the formula

$$T=\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum in feet, and [illegible] 32.2. Find the time of the *beat* of a pendulum 2 feet long, to with[illegible] tenth of a second.

4. The flow in cubic feet per second of water [illegible] a trapezoidal weir is given by the formula

$$q=3.3\,Lh^{3/2},$$

where L is the length of the weir in feet, and h t[illegible]head of water in feet on the weir. How many cubic feet of water pe[illegible]cond will flow over a weir 30 feet long and with a head of two feet of [illegible]ter on the weir?

5. The rate at which a principal P will amount to S when interest is compounded annually for n years is

$$i = \sqrt[n]{\frac{S}{P}} - 1.$$

What is the rate when \$535 placed at compound interest for 6 years amounts to \$696.71?

6. When interest is compounded m times a year the *nominal* rate j corresponding to the *effective* rate i is given by the formula

$$j = m\{(1+i)^{\frac{1}{m}} - 1\}.$$

What is the nominal rate corresponding to 4 % (= .04) when interest is compounded quarterly?

7. The amount A due a man who deposits P dollars every quarter for n years in a savings bank paying interest at rate i is

$$A = P\frac{(1+i)^n - 1}{(1+i)^{1/4} - 1}.$$

[illegible] uthhat [illegible]e the amount of the savings of a man who deposits \$50 each better a[illegible]ter [illegible] years, if the bank pays 4 %?

8. Gi[illegible]irections by which a carpenter can cut a board of length $2 + \sqrt{[illegible]}$ ft[illegible]

32. F[illegible]mental Theorem concerning Radicals. *If* $a + \sqrt{b} =$ [illegible] $\sqrt{y}$, *where* a, b, x, *and* y *are rational and* $\sqrt{b}$ *and* $\sqrt{x}$ *are i*[illegible]*al, then* $a = b$ *and* $x = y$.

Proof. [illegible]en x is transposed to the left member, the equality

$$a + \sqrt{b} = x + \sqrt{y},$$

becomes

$$-x + \sqrt{b} = \sqrt{y}.$$

Squaring both [illegible]s and transposing the rational parts to the right member, [illegible]ind

$$2[illegible]- x)\sqrt{b} = y - b - (a - x)^2.$$

From this equatio[illegible] is clear that $a - x$ must be zero. Other-

wise, the irrational number $2(a-x)\sqrt{b}$ is equal to a rational number. But if $a-x=0$, then $a=x$; and from the right member $y-b=0$, or $y=b$. Q.E.D.

COROLLARY. *If* $P+\sqrt{R}=0$, *then* $P=0$ *and* $R=0$.

33. Imaginary Numbers. A simple radical like $\sqrt{a}$ is squared by removing the radical sign. Thus $(\sqrt{5})^2=5$. Similarly, $(\sqrt{-5})^2=-5$. In the second example it is seen that $\sqrt{-5}$ is a number whose square is negative. Hence it cannot be either a positive or a negative number in the ordinary sense, since the squares of both positive and negative numbers are positive.

A number whose square is negative is called an ***imaginary number***, or more accurately, a ***pure imaginary number***. A number whose square is positive, or zero, is a ***real number***. The sum of a real and a pure imaginary number is called a ***complex number***. Thus, 5, $-\sqrt{5}$, $2+\sqrt{5}$ are real numbers, $\sqrt{-5}$, $\sqrt{-1}$ are pure imaginary numbers, while $2+\sqrt{-5}$, $2-\sqrt{-5}$ are complex numbers. All such numbers constitute the ***complex number system*** which includes the real as well as the pure imaginary numbers.

Computation with imaginary numbers is like computation with real numbers with a single important exception, namely, *the product of two pure imaginary numbers both of which are preceded by the positive sign is a negative number.* For example,

$$\sqrt{-5}\times\sqrt{-7}=-\sqrt{35}.$$

Similarly,

$$(-\sqrt{-5})\times\sqrt{-7}=\sqrt{-5}\times(-\sqrt{-7})=+\sqrt{35}.$$

This apparent difference is easily explained by going back to fractional exponents. Thus,

$$\begin{aligned}\sqrt{-5}\times\sqrt{-7}&=[5\times(-1)]^{1/2}\times[7\times(-1)]^{1/2}\\&=5^{1/2}(-1)^{1/2}7^{1/2}(-1)^{1/2}=35^{1/2}[(-1)^{1/2}]^2=-35^{1/2}.\end{aligned}$$

More generally, if the bases are negative,

$$(-a)^{1/2}(-b)^{1/2} = a^{1/2}b^{1/2}(-1)^{1/2}(-1)^{1/2}$$
$$= (ab)^{1/2}(-1)^{2/2} = -(ab)^{1/2}.$$

This is an illustration of the exceptional case pointed out in the footnote in § 25.

In elementary problems the occurrence of imaginaries denotes impossibility. (See §§ 65 and 77.)

EXERCISES

1. Perform the operations indicated in the following.

(a) $\sqrt{-3} \times \sqrt{-7}$.
(b) $-\sqrt{-2} \times \sqrt{-5}$.
(c) $3\sqrt{-5} \times 4\sqrt{-7}$.
(d) $(3 + 2\sqrt{-2})^2$.
(e) $(3 + 2\sqrt{-2})^3$.
(f) $(3 + \sqrt{-5})(3 - \sqrt{-5})$.
(g) $\sqrt{-2}(3 + 2\sqrt{-5})$.
(h) $\sqrt{-2}(\sqrt{2} + 2\sqrt{-3})$.
(i) $\sqrt{-4} + \sqrt{-36} + \sqrt{-25}$.

2. Reduce the fraction

$$\frac{2 + \sqrt{-3}}{3 + \sqrt{-5}}$$

to an equivalent fraction whose denominator is real, and express the result in the form $A + B\sqrt{-1}$ where A and B are real, though not necessarily rational, numbers.

3. Find the base of a right triangle whose altitude is 7 and hypotenuse is 5.

CHAPTER IV

LOGARITHMS

34. Definitions. In books on analysis it is proved that every real positive number y can be expressed as a power of a real base a which is different from 1 and from 0; that is

$$(1) \qquad y = a^x.$$

In this equation x is called the ***logarithm of y to the base a,*** and is written

$$(2) \qquad x = \log_a y.$$

For present purposes, equations (1) and (2) mean exactly the same thing and may be used interchangeably.

The definition of a logarithm stated in words is as follows. *The logarithm of a number is the exponent that indicates the power to which a base must be raised in order to produce the given number.*

35. Theorems relating to Logarithms. Since logarithms are exponents, every theorem about exponents is also a theorem about logarithms when expressed in the language of logarithms. Theorems relating to logarithms will therefore be proved by direct reference to the properties of exponents.

THEOREM 1. *The logarithm of unity is zero.*

For, if we put $y = 1$ in equation (1), x must equal zero, since $1 = a^0$. (See equation (1), § 23.) It follows that

$$(3) \qquad \log_a 1 = 0.$$

THEOREM 2. *The logarithm of the base is unity.*

For, if we put $y = a$ in equation (1), x must be 1, and the equation $a = a^1$ means, in the language of logarithms,

$$\textbf{(4)} \qquad \log_a a = 1.$$

THEOREM 3. *The logarithm of the product of two numbers is equal to the sum of their logarithms.*

If y_1 and y_2 be the numbers and x_1 and x_2 their logarithms, the definition gives

$$y_1 = a^{x_1} \quad \text{and} \quad y_2 = a^{x_2},$$

whence, by I, § 22,

$$y_1 y_2 = a^{x_1 + x_2}.$$

In the language of logarithms, the last equation gives

$$\log_a y_1 y_2 = x_1 + x_2.$$

But since

$$x_1 = \log_a y_1 \quad \text{and} \quad x_2 = \log_a y_2,$$

it follows that

$$(5) \qquad \log_a y_1 y_2 = \log_a y_1 + \log_a y_2,$$

as was to be proven.

THEOREM 4. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the denominator.*

Let y_1 be the dividend and y_2 the divisor, x_1 the logarithm of y_1, and x_2 the logarithm of y_2. By definition

$$y_1 = a^{x_1} \quad y_2 = a^{x_2};$$

whence

$$\frac{y_1}{y_2} = a^{x_1 - x_2}.$$

In the language of logarithms the last equation gives

$$\log_a \frac{y_1}{y_2} = x_1 - x_2,$$

or

$$\log_a \frac{y_1}{y_2} = \log_a y_1 - \log_a y_2, \tag{6}$$

which was to be proven.

THEOREM 5. *The logarithm of a power is equal to the logarithm of the base of the power multiplied by the exponent of the power.*

For, if

$$y = a^x$$

be the number, and m the exponent, then by III, § 22,

$$y^m = (a^x)^m = a^{mx}.$$

From the last equation,

$$\log_a y^m = mx,$$

or

$$\log_a y^m = m \log_a y. \tag{7}$$

COROLLARY. *If $m = 1/n$, where n is an integer, equation* (7) *takes the form*

$$\log_a \sqrt[n]{y} = \frac{1}{n} \log_a y. \tag{8}$$

Translated into words, equation (8) says that *the logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

36. Systems of Logarithms. The logarithms of all real positive numbers with a given number as a base form a so-called ***system of logarithms.*** On account of its great convenience, the system whose base is 10 is used in all numerical computation. The system whose base is 10 is called the ***common*** or ***Briggsian*** system.

Unless otherwise stated, common logarithms will be used and all reference to the base will be omitted. For example, we shall write log 25, not $\log_{10} 25$.

37. Characteristic and Mantissa. A table of powers of 10 may be written in exponential and logarithmic forms, as follows.

Exponential Form	Logarithmic Form
. . . . . .	. . . .
$10^{-6} = .000001$	$\log .000001 = -6$
$10^{-5} = .00001$	$\log .00001 = -5$
$10^{-4} = .0001$	$\log .0001 = -4$
$10^{-3} = .001$	$\log .001 = -3$
$10^{-2} = .01$	$\log .01 = -2$
$10^{-1} = .1$	$\log .1 = -1$
$10^{0} = 1.$	$\log 1 = 0$
$10^{1} = 10$	$\log 10 = 1$
$10^{2} = 100$	$\log 100 = 2$
$10^{3} = 1000$	$\log 1000 = 3$
$10^{4} = 10{,}000$	$\log 10{,}000 = 4$
$10^{5} = 100{,}000$	$\log 100{,}000 = 5$
$10^{6} = 1{,}000{,}000$	$\log 1{,}000{,}000 = 6$
. . . .	. . . .

The logarithm of an integral power of 10 is an integer, either positive or negative. The logarithm of a number which is not an integral power of 10 lies between two integers and must be expressed, either exactly or approximately, as an integer *plus* a decimal part. For example, the preceding table of powers shows that the integral part of log 258 is 2 because 258 lies between 100 and 1000, and its logarithm must therefore lie between 2 and 3. From the table of logarithms (Table, p. 252) the remaining part is found to be approximately .4116. We may then write

$$\log 258 = 2.4116.$$

The integral part of a logarithm is called the ***characteristic***, and the decimal part is called the ***mantissa***.

Even when the logarithm is negative it may be so written

that the mantissa is positive. For example, it is known that approximately,

$$\log .025 = -1.6021.$$

When 10 is added and subtracted, this equality may be written in the form

$$\log .025 = 10 - 1.6021 - 10 = 8.3979 - 10.$$

In the last form the mantissa is the *positive* number .3979, and the characteristic is the negative number $8 - 10$, or -2.

In what follows it will always be assumed that *the logarithm is written in such a form that the mantissa is positive.*

THEOREM 6. — *The mantissas of the common logarithms of all numbers having the same sequence of figures are equal regardless of the position of the decimal point.*

PROOF. Moving the decimal point is equivalent to multiplying or dividing the number by a power of 10. The logarithm would be changed by adding or subtracting the logarithm of this power of 10. But the logarithm of a power of 10 is an integer, and adding or subtracting an integer cannot change the mantissa.

EXAMPLE.
$$\begin{aligned}\text{Log } 2.58 &= \log \frac{258}{100} = \log 258 - \log 100 \\ &= 2.4116 - 2 \\ &= 0.4116\end{aligned}$$

38. Determination of the Characteristic. — From the table in § 37, it is clear that the characteristic of the logarithm of 1, or of any number between 1 and 10, is zero. Moving the decimal point one place to the right is equivalent to multiplying the number by 10, and therefore, by Theorem 3, to increasing its logarithm by 1. On the other hand, moving the decimal point one place to the left is equivalent to dividing the number

by 10, and, by Theorem 4, to diminishing the characteristic by 1. It follows that *the characteristic of the logarithm of any number may be found by comparing the position of its decimal point with that of another number having the same sequence of figures and lying between* 1 *and* 10.

Thus to find the characteristic of log 2580 it is sufficient to note that $2580 = 10^3 \times 2.58$. The characteristic is therefore 3.

Similarly the characteristic of log .000258 is -4, since

$$.000258 = 10^{-4} \times 2.58,$$

In general, the *characteristic is equal to the exponent of that power of* 10 *which must be used as a multiplier to produce the given number from a number having the same sequence of figures and lying between* 1 *and* 10.

An easy way to determine the correct characteristic is first to imagine the decimal point placed after the first significant digit (not zero) and then to count forward or backward to the actual decimal point. The number of digits passed over is the characteristic, counted positive if the motion was to the right and counted negative if the motion was to the left.

A negative characteristic is written on the right of the mantissa, or, it is broken up into two parts, one of which is a multiple of -10, and this multiple of -10 is written on the right of the mantissa. Thus,

$$\log .000258 = .4116 - 4 = 6.4116 - 10,$$

since the characteristic is -4.

EXERCISES

1. Using the table, find the logarithms of each of the following numbers.

(*a*) 2.34. (*d*) .359. (*g*) $\dfrac{35.6}{28.4}$.

(*b*) .234. (*e*) .0293. (*h*) $\dfrac{35.6}{.0438}$.

(*c*) .000234. (*f*) $(.0346)^2$. (*i*) $\left(\dfrac{.392}{2.14}\right)^3$.

39. Interpolation. Logarithms, like square and cube roots, can be expressed only approximately in the ordinary notation. The tables in common use give the mantissas to four, five, or six decimal places, leaving the computer to supply the characteristic according to the rule of § 38.

By means of a four-place table the logarithm of a three-figure number may be read off directly, but the logarithm of a four-figure number must be found by *interpolating* a logarithm between two logarithms of the table. For example, suppose the logarithm of 258.3 is wanted. Clearly log 258.3 lies between log 258 and log 259. From the table

$$\log 258 = 2.4116, \quad \text{and} \quad \log 259 = 2.4133.$$

The difference between these two mantissas is .0017. The difference between 258 and 258.3 is 3 tenths of the difference between 258 and 259. The difference between log 258 and log 258.3 is therefore 3 tenths of .0017, or .0005; and we write $\log 258.3 = \log 258 + .0005 = 2.4121$.

40. Multiplication and Division of Logarithms. A logarithm whose characteristic is positive is multiplied and divided in the same manner as any positive number is multiplied or divided. If the characteristic is negative the operation of division requires some care. For example, to divide $.4126 - 4$ by 7, the logarithm should be written so that the negative part is exactly divisible by 7. Thus $.4126-4=3.4126-7=66.4126-70$. Consequently, one seventh of $.4126 - 4$ is $.4875 - 1$, or $9.4875 - 10$. Either result is in convenient form.

EXERCISES

Find the logarithms of each of the following numbers.

1. 534.2.

2. 534.2617.

3. 6.278.

4. .05342.

5. $(.3826)^5$.

6. $(3.826)^5$.

7. $\sqrt{.3427}$.

8. $\sqrt[3]{.02796}$.

9. $(.03562)^{3/2}$.

10. $8^{\frac{1}{4}}$.

11. $8^{\frac{1}{5}}$.

12. $\left(\frac{34.26}{28.16}\right)^4$.

41. Antilogarithms. In order to complete a computation by logarithms it is necessary to find from the table the ***antilogarithm***, that is, the number corresponding to a given logarithm. To illustrate, suppose

$$\log x = 2.5432$$

is given, and the number x is required. The nearest mantissa given by the table is .5428 which corresponds to the number 349. The difference between this mantissa and the given mantissa is .0004, while the difference between the mantissa .5428 and the next mantissa in the table is .0013. The difference in number is therefore $\frac{4}{13}$ of 1, or, to the nearest tenth, .3. The first four digits in the required number are then 3493. From the relation between decimal point and characteristic as given in § 38, the point must lie between the 9 and the 3 so that

$$x = 349.3.$$

If $\log x = 8.54325 - 10$, then $x = .03493$.

42. Cologarithms. The direct subtraction of logarithms may be avoided by the use of ***cologarithms***. The cologarithm of a number is the negative of the logarithm, *i.e.* the logarithm of the reciprocal of the number. It is written in the form

$$\text{(8)} \qquad \operatorname{colog} n = \log \frac{1}{n} = -\log n = 10 - \log n - 10,$$

where 10 has been added to and subtracted from $-\log n$. For example,

$$\begin{aligned} \operatorname{colog} 258 = \log \tfrac{1}{258} &= 10 - 2.4116 - 10 \\ &= 7.5884 - 10. \end{aligned}$$

$$\begin{aligned} \text{Again, } \operatorname{colog} .0258 = \log \frac{1}{.0258} &= 10 - (8.4116 - 10) - 10 \\ &= 1.5884. \end{aligned}$$

In finding the cologarithm from the logarithm, expert computers begin the subtraction on the left, taking every figure up

to, but not including, the last significant figure from 9. For example,

$$\text{Log}\,\frac{34.58 \times .01237}{2.159 \times .1673}$$
$$= \log 34.58 + \log .01237 + \text{colog}\,2.159 + \text{colog}\,.1673$$
$$= 1.5388 + 8.0924 - 10 + 9.6658 - 10 + 0.7765$$
$$= 0.0735.$$

MISCELLANEOUS EXERCISES

1. Compute the value of each of the following expressions by means of logarithms.

(a) $259 \times 342 \times .0621.$

(b) $\dfrac{8394 \times 5728}{9342}.$

(c) $\dfrac{3782 \times \sqrt[7]{3849}}{9234}.$

(d) $\dfrac{2594 \times .8431}{\sqrt[3]{3241}}.$

(e) $\dfrac{.8432 \times \sqrt[7]{.02315}}{.03421}.$

(f) $\sqrt[7]{\left(\dfrac{5294}{6382}\right)^3}.$

(g) $\dfrac{\sqrt[7]{.03428}}{\sqrt[11]{.03795}}.$

(h) $5.34^{.021}.$

(i) $631^{-.54}.$

2. Compute the value of

$$\frac{(-5347)^2(\sqrt[3]{-59.32})}{(-76843)^3}.$$

[HINT. Logarithms will give only the absolute value of the result. The sign must be determined independently.]

3. Find the area of a circle whose radius is 36.59.

4. Find the volume of a sphere whose radius is 2.634.

5. Find the volume of a right circular cone whose altitude is 5.374 and the radius of whose base is 2.634.

6. Find the simple interest on $239.26 at 5.75 % for 2 years and 9 months.

7. Find the compound amount of $ 1225.76 for 4 years with interest at 6 % compounded semiannually, given that the formula for the compound amount is

$$P\left(1+\frac{r}{p}\right)^{np},$$

where P is principal, r rate, n the time expressed in years, and p the number of times the interest is compounded each year.

Would the result obtained by means of four-place tables be acceptable in commercial practice?

8. A man invested $ 5000 in business and at the end of 5 years drew out $ 7439.62. What rate at compound interest payable annually did he receive on the investment?

9. The weight of a cubic foot of water is 1000 ounces. What is the weight in tons of a ship whose "displacement" is 59,200 cubic feet?

10. The time in seconds of one complete oscillation of a pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is measured in feet and $g = 32.2$. What is the time of a complete oscillation of a pendulum 27 inches long?

11. One meter is equivalent to 39.37 inches. How many inches are there in 1275 centimeters?

12. Reduce 435 square centimeters to square inches.

13. Reduce 579 square inches to square centimeters.

14. Ordinary interest is figured on the basis of 360 days and exact interest on the basis of 365 days to the year. The relation between the two is given by the formula

$$\textit{exact interest} = \textit{ordinary interest} \times \tfrac{72}{73}.$$

Find by a single computation the exact interest on $ 532.75 for 139 days at 6 %.

15. When air expands without gain or loss of heat the pressure is approximately given by the formula

$$p = c\rho^{1.41}$$

where ρ is density and c is a constant. Find the value of p when $c = 566.9$ and $\rho = .0026$.

43. The Logarithmic Scale and the Slide Rule. The logarithms of the integers 1 to 10 inclusive are, if we use two decimal places only,

.00, .30, .48, .60, .70, .78, .85, .90, .95, 1.00.

These numbers may be plotted easily upon any line segment one unit in length and divided into hundredths. If we attach to each point, not its logarithm, but the corresponding number we obtain a so-called ***Gunter's*** or ***logarithmic scale.*** (See Fig. 4.)



Fig. 4.

A ***slide rule*** consists essentially of two logarithmic scales A and B which may be moved over each other as in Fig. 5.

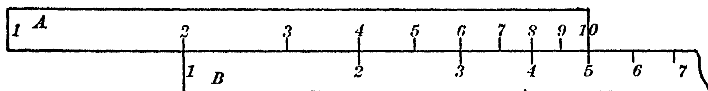


Fig. 5.

To multiply two numbers, as 2 and 4, by means of a slide rule, we set the 1 of scale B upon the 2 of scale A and count *forward* to 4 on scale B. We have in this way that point on scale A which gives the sum of log 2 and log 4. The point is of course 8.

Similarly, if we wish to divide 8 by 4 we would set 4 of scale B upon 8 of A then count *back* on A to 1 of B. This point denotes the difference between log 8 and log 4.

Clearly the logarithmic scale gives only the mantissas of the logarithms. The determination of the characteristics presents no difficulty, at least in the simpler cases.

The slide rule in actual use is considerably more elaborate than the pair of logarithmic scales in Fig. 5. The illustration

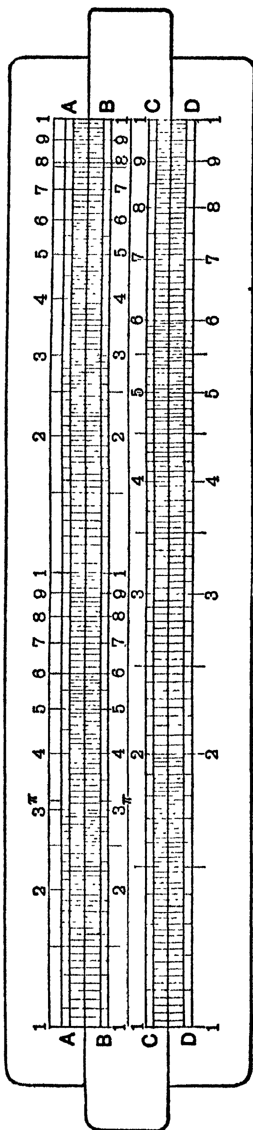


Fig. 6.

(Fig. 6), which is a reproduction from a photograph of an actual slide rule, shows two pairs of scales. The upper pair consists of two logarithmic scales placed end to end.

By means of a ten-inch rule products and quotients, true to two or three figures, may be found with surprising rapidity. The longer the rule the greater the accuracy.

EXERCISES

1. Perform each of the following multiplications by means of a slide rule.

(*a*) 3×2. (*b*) 3.5×2.5. (*c*) 2.7×3.4. (*d*) 5×4.

(*e*) 5.5×4.5. (*f*) 5.6×7.3. (*g*) 1.45×23. (*h*) 14.5×2.3.

2. Perform each of the following divisions by means of a slide rule.

(*a*) $6 \div 3$. (*b*) $6.5 \div 3.5$. (*c*) $30 \div 5$.

(*d*) $32.5 \div 4.6$. (*e*) $21.4 \div 63.5$. (*f*) $4.52 \div 0.125$.

3. Perform each of the following indicated operations by means of a slide rule.

(*a*) $3 \times 4 \times 5$. (*b*) $3.7 \times 4.5 \times 5.4$. (*c*) $6.9 \times 11 \div 4.5$.

(*d*) $(45)^2 \div 63$. (*e*) $\pi(4.52)^2$. (*f*) $4\pi(68.2)^2$.

CHAPTER V

FUNCTIONS OF A VARIABLE — GRAPHICAL REPRESENTATION

44. Definitions. If a man walks at the uniform rate of 3 miles per hour the distance covered in x hours is $3x$ miles. If y denotes the distance, the relation between distance and time is given by the equation

$$y = 3x.$$

In this example x may represent *any* arbitrary number. Moreover to every value of x there corresponds a definite value of y. Numbers like x and y in this example, which assume many values in the same discussion, are called ***variables.*** The important fact to be noted in the example is that when the value of one variable x is given the value of the other is known.

If two variables, x and y, are so related that when the value of x is given the value of y is known, then y is said to be a ***function*** *of x.*

The notion of *dependence* of one variable upon the other is fundamental. The variable which is assumed to take arbitrary values is called the ***independent variable.*** The variable whose values depend upon the values of the independent variable is called the ***dependent variable.*** In the example just given it has been assumed that the values of x are arbitrary. Therefore x is the independent, and y the dependent variable.

The notion of a function is one of the most important ideas with which mathematics has to do. The following examples will serve to illustrate this statement.

EXAMPLE 1. If $100 be loaned at 6 % the interest is a function of the time, which is expressed by

$$I = \$100 \times .06 \times t.$$

EXAMPLE 2. The distance through which a body falling freely from rest will travel in time t, is a function of the time given by

$$s = \frac{1}{2} gt^2,$$

where g is a constant.

EXAMPLE 3. The circumference C and the area A of a circle are functions of the length of the radius given by

$$C = 2 \pi r,$$

and

$$A = \pi r^2,$$

respectively.

EXAMPLE 4. If x denote any number and y its logarithm, the functional relation

$$y = \log x$$

expresses the relation between the numbers of the number column and the corresponding logarithms of a table of logarithms.

EXAMPLE 5. If y denote the angle which a stairway with 12-inch treads makes with the floor, and if h denote the height of the riser, the steepness of the stairway depends upon the height of the riser and is expressed as the *ratio of riser to tread.* In trigonometry this ratio is called the ***tangent*** of the angle y, so that by definition,

$$\tan y = \frac{h}{12}.$$

It should be carefully noted that in the examples just given,

$$I, \quad s, \quad C, \quad A, \quad y, \quad \text{and} \quad \tan y$$

are merely symbols for the functions

$$100 \times .06 \times t, \quad \frac{gt^2}{2}, \quad 2\pi r, \quad \pi r^2, \quad \log x, \quad \text{and} \quad \frac{h}{12},$$

respectively. All these functions are given by definite mathe-

matical expressions, a thing which is by no means necessary to make one variable a function of another.

When it is desired to express the fact that a variable y is a function of x without stating the exact mode of dependence, it is customary to write

$$(1) \qquad y = f(x),$$

which is read "y equals a function of x," or "y is a function of x."

EXERCISES

1. One leg of a right triangle is x and the other is 5. Express the length of the hypotenuse as a function of x.

2. Let x denote the side of an equilateral triangle. Express the area as a function of x.

3. Let a denote the radius of a circle and let x denote the distance from the center of the circle to a chord. Express the length of the chord as a function of x.

4. Express the side of a square as a function of the area.

5. Express the area of an equilateral triangle as a function of the length of a side.

6. Express the side of an equilateral triangle as a function of the area.

7. Express the simple interest on one dollar as a function of the time t when the rate is 5%; as a function of the rate r, when the time is 10 years.

8. Express the amount of one dollar at compound interest as a function of the time t when the rate is r.

9. At a given time a man 12 miles from a given place begins to travel away from the place at the uniform rate of 12 miles per hour. What function expresses his distance from the place at the end of t hours?

10. In Ex. 9, suppose the man begins to travel toward the place. What is the function?

11. A man starts from a point 3 miles east of a certain place and travels east at the rate of 4 miles per hour. What function expresses his distance from the place at the end of x hours?

45. Coördinates. The independent variable may take an unlimited number of values. To a given value of the independent variable corresponds one value (or several) of the function. This pair of values, one for the independent variable and one for the function, may be used to determine the position of a point in a plane in the following manner.

Let the function be $x+2$. To the value $x=2$ corresponds the value $y=4$. The value $x=2$ can be represented by a line OA two units long laid off to the right of the arbitrary point O in the arbitrary line XX' (Fig. 7). Similarly the value $y=4$ can be represented by a line 4 units long laid off upward from A, parallel to a line drawn through O, and perpendicular to the line XX'.

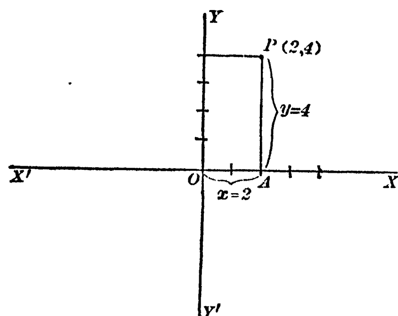


FIG. 7.

The arbitrary line XX', in which the values of x are measured, is called the ***x-axis***, or the ***axis of the variable***, while the line YY', parallel to which the values of the function are drawn, is called the ***y-axis***, or the ***function-axis***. The pair of values $x=2$ and $y=4$ determine a point which is found by measuring 2 units along the x-axis and then 4 units upward parallel to the y-axis. (See Fig. 7.)

If either x or y is negative the corresponding line is drawn in the opposite direction. Thus $x=-2$ and $y=4$ determine a point two units to the left of the y-axis and four units above the x-axis.

The distance measured along the x-axis is called the ***abscissa*** and the distance measured along the y-axis is called the ***ordinate***. The two together are called the ***coördinates*** of the point

which they determine. The point in Fig. 7 is indicated by the symbol (2, 4). In general, we write (x, y) where x is the abscissa and y the ordinate.

46. The Graph of a Function. By giving arbitrary values to x, any number of values of the function may be found. The relation between the variable and the function may be exhibited as in the following table, which has been constructed for the function $x + 2$.

x	-4	-3	-2	-1	0	1	2	3	4
$x+2$	-2	-1	0	1	2	3	4	5	6

If a pair of axes be drawn, each value of x with the corresponding value of the function determines a point which is easily located by the method of § 45. The lines $1\,P_1$, $2\,P_2$, $3\,P_3$, ⋯ represent the values of the function $x + 2$ for the corresponding values 1, 2, 3, ⋯ of x. (See Fig. 8.)

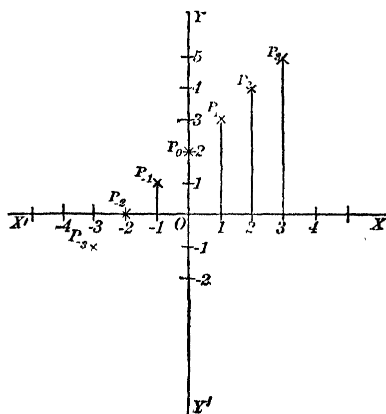


Fig. 8.

If the points on the x-axis be indefinitely near together, the points P will be indefinitely near together and will lie in a line, either curved or straight. The line thus determined is called the ***graph of the function.*** The graph of a function is constructed by locating a sufficient number of its points and then drawing a smooth curve through these points.

47. Coördinate Paper. By means of paper ruled in small squares, all necessity for measuring instruments to determine

the lengths of coördinates is avoided, since the spaces on the paper serve as a measuring scale. Paper so ruled is called ***coördinate paper***, or ***squared paper***. Coördinate paper with 8 or 10 rulings to the inch, or 5 rulings to the centimeter, is most convenient.

EXERCISES

1. Mark on coördinate paper the points $(1, 1)$, $(2, 3)$, $(-1, 3)$, $(-1, 0)$, $(0, 1)$, $(-3, -2)$, $(2, -3)$, $(0, 0)$.

2. Construct the graphs of the following functions.

(*a*) $2x$.	(*e*) $\frac{1}{3}x$.	(*i*) $\frac{1}{2}x + 2$.
(*b*) $3x$.	(*f*) $\frac{1}{4}x$.	(*j*) $\frac{1}{2}x - 2$.
(*c*) x.	(*g*) $x - 2$.	(*k*) $3x + 1$.
(*d*) $\frac{1}{2}x$.	(*h*) $2x - 2$.	(*l*) $3x - 1$.

3. What is your conjecture concerning the geometric character of the graphs in Exs. 2 (*a*) – 2 (*l*)?

4. What is the relation between the graphs in Exs. 2 (*a*) and 2 (*h*), and in Exs. 2 (*b*), 2 (*k*), and 2 (*l*)?

5. What is the relation between the graphs in Exs. 2 (*d*), 2 (*e*), and 2 (*f*)? In general, what effect does the size of the coefficient of x seem to have upon the graph?

6. Construct the graph determined by $y = 4x^0$.

7. Which of the points $(-1, 1)$, $(1, 1)$, $(0, 3)$, $(2, 4)$, $(1, 3)$, $(1, 5)$ lie on the graph of the function

$$y = 2x + 3\,?$$

48. The Linear Integral Function. A function of the form

$$ax + b, \tag{2}$$

where a and b are constants, is called a ***linear integral function***.

The graphs of the linear integral functions in Exs. 2–7 of the preceding section are apparently all straight lines. That the graph of every linear integral function is a straight line may be proven as follows.

Let y denote the function; then

(3) $$y = ax + b.$$

When $x = 0$, $y = b$, so that the graph passes through the point $(0, b)$. (See Fig. 9, which has been drawn for the case in which a and b are both positive.) Now, if x is increased, the increase in y is a times as great. When x changes from $x = 0$

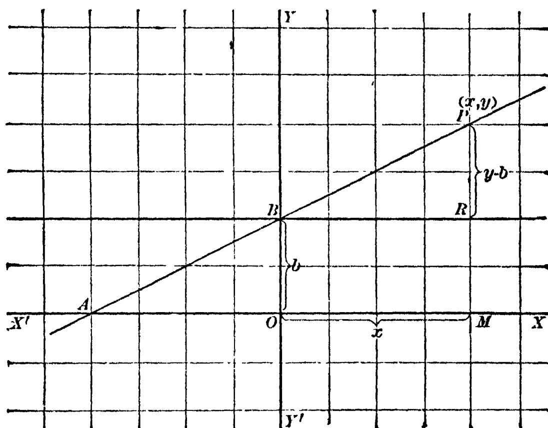


FIG. 9.

to $x = BR$, the corresponding change in y is RP. It follows therefore that

$$RP = a \cdot BR,$$

or

(4) $$\frac{RP}{BR} = a.$$

Equation (4) expresses the fact that the point P moves in such a way that the ratio of its distances from two fixed lines is constant. From geometry we know that the locus of a point the ratio of whose distances from two fixed lines is constant is a straight line. Q.E.D.

49. The Graph of a Linear Equation. The straight line in Fig. 9 may be looked upon, either as the graph of the function

$$ax + b,$$

or as the graph of the equation

$$y = ax + b. \tag{5}$$

Moreover, every equation of the first degree with *two* variables x and y, can be put in the form (5). For, if the equation

$$Ax + By + C = 0 \tag{6}$$

be divided through by B, it becomes, after transposition,

$$y = -\frac{A}{B}x - \frac{C}{B}. \tag{7}$$

Equation (7) is identical with (5) if

$$-\frac{A}{B} = a \text{ and } -\frac{C}{B} = b.$$

It follows from what has been said that *the graph of every equation of the first degree between two variables x and y is a straight line.* For that reason such an equation is called a ***linear equation.****

When a linear equation is written in the form (5), the number a is called the ***slope*** of the line. The reason for this name is given by equation (4), which shows that the steepness of the line is determined by a.

The slope of a line may be likened to the *grade* of a hill or to the *steepness* of a stairway, which is expressed as the ratio of riser to tread. (See Ex. 5, § 44.) It is also identical with that trigonometric function of the angle RBP (Fig. 9) which is called the *tangent* of the angle RBP. Clearly two lines having the same slope with respect to the x-axis are parallel.

* The statement is true even though A or B is zero and the equation reduces to a linear equation in one variable.

The two line-segments OA and OB (Fig. 9), or their lengths, are called the ***x-intercept*** and the ***y-intercept*** respectively. The x-intercept is the value of x corresponding to the value $y = 0$ and the y-intercept is the value of y corresponding to the value $x = 0$.

Consequently, the intercepts are

$$OB = b, \text{ and } OA = -\frac{b}{a}.$$

Since the graph of the linear integral function, or of the linear equation, is known to be a straight line, the line may be constructed as soon as two of its points are known. Usually the points determined by the intercepts are most easily found.

50. Uniform Motion. If a body moves in such a way that it travels over equal spaces in equal times, we say that its motion is ***uniform.*** The measure of the space traversed in a unit of time is called the ***velocity***, or the ***speed.*** For uniform motion the speed is a constant number, depending, however, upon units employed for time and for space. For example, a speed of 60 miles per hour is the same as one mile per minute, or 88 feet per second.

The space s traversed in time t is a function of the time, and if the speed v is a constant, s is given by the equation

$$(8) \qquad s = vt.$$

In the most general case of uniform motion the *space is a linear integral function* of t (see equation (9) below), and it is, therefore, represented graphically by a straight line with slope v. The greater the velocity, the steeper the line.

EXERCISES

1. Construct the graph of each of the following functions.

(*a*) $3x - 5$.
(*b*) $x/2 - 3$.
(*c*) $-2x + 1$.
(*d*) $-3x + 1$.
(*e*) $\sqrt{2}x + 1$.
(*f*) $\sqrt{2}x - 1$.

2. Construct the graph of each of the following equations, and find the intercepts and the slope in each case.

(a) $y = -x + 2$.
(b) $y = -2x + 1$.
(c) $y = \sqrt{3}x + 1$.
(d) $2x + 3y = 5$.
(e) $2y - 3x = 5$.
(f) $2x + 3y + 5 = 0$.

3. Construct the graph of the equation $y - 1 = 0$.

4. Construct the graph of the equation $x - 2 = 0$.

5. Hooke's Law for the extension of a stretched spring is

$$E = kt,$$

where E is the extension due to the tension, or pull, t. Represent the extension graphically when $k = 5$.

6. From the equation

$$c = \pi d,$$

construct a graph from which you can read off the length of the circumference of any circle whose diameter is known.

7. The formula for simple interest is

$$I = Prt,$$

where P is principal, r rate of interest, and t the time expressed in years. Construct on a single sheet of coördinate paper the graphs

(a) when $t = 1$ and $r = .06$;
(b) when $t = 2$ and $r = .06$;
(c) when $t = 3$ and $r = .06$.

Show how these graphs may be used instead of an interest table for 6%.

8. If C and F represent the readings on Centigrade and Fahrenheit thermometers, the relation between the two is given by

$$F = \tfrac{9}{5} C + 32°.$$

Construct the graph of the equation and demonstrate its use.

9. A man travels at the uniform rate of 30 miles per hour. Construct the graph which shows the distance traveled in time t.

[Hint. Let 10 miles represent one unit on the distance axis.]

10. A man travels toward his home 100 miles away at the rate of 30 miles per hour. What function represents his distance from home at the time t? Construct the graph.

11. The condensed schedule of a certain train running between Chicago and Madison is as follows:

Miles	Stations	Time
0	Chicago	4.00 P.M.
41.1	Gray's Lake	5.03 P.M.
74.6	Walworth	6.03 P.M.
99.2	Janesville	7.05 P.M.
139.7	Madison	8.25 P.M.

Assuming the speed to be constant between the stations given, construct the graph showing distances from Chicago at times t. Determine from the graph what part of the trip is made in the fastest time.

12. A body moving with uniform velocity of v feet per second passes a point s_0 feet from A at time t_0. Show that the distance s from A at time t is given by the equation

$$(9) \qquad s - s_0 = v(t - t_0).$$

13. A train leaving Pittsburgh at 12 : 25 P.M. passes Steubenville 43.2 miles from Pittsburgh at 1 : 41 P.M and Denison 90.5 miles from Pittsburgh at 2 : 50 P.M. Find the equation of motion giving the distance of the train from Pittsburgh, assuming that the speed is constant.

14. A train traveling toward Pittsburgh at the rate of 38.5 miles per hour passes through Steubenville 43 2 miles from Pittsburgh at 3 : 00 A.M. Find the equation that will express the distance from Pittsburgh in terms of the time the train has been on its way.

15. Construct a straight line which will represent graphically the motion of a body moving uniformly at the rate of 2 feet per second. Is there more than one line that satisfies the requirement? Explain.

16. A train does not move uniformly between stations but starts slowly and stops gradually. Sketch a graph that corresponds roughly to the facts.

51. The Rational Integral Quadratic Function. A function which has the form

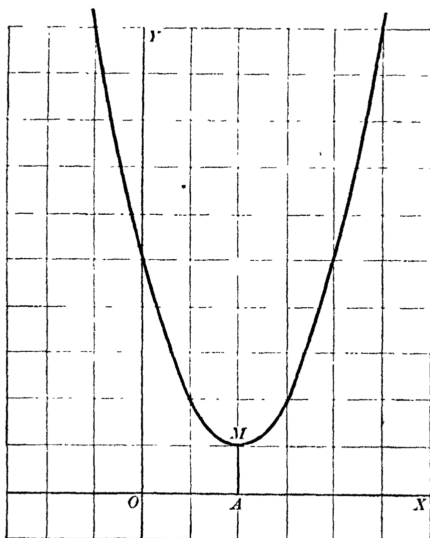
$$(10) \qquad ax^2 + bx + c,$$

where a, b, and c are constants, is called a ***rational integral quadratic function.***

The graph of a quadratic function is constructed by making a table of values as in the case of linear functions, and then drawing a curve through the points thus determined. For example, the table for the function $x^2 - 4x + 5$ is in part as follows:

x	-1	0	1	2	3	4
$f(x)$	10	5	2	1	2	5

Drawing a line through the points $(-1, 10)$, $(0, 5)$, $(1, 2)$, $(2, 1)$, $(3, 2)$, $(4, 5)$, we obtain the curve shown in Fig. 10.



FIG

The graphs of all quadratic functions for which the coefficient of x^2 is positive resemble in form and in position the curve in Fig. 10. The curves may or may not cut the x-axis, but in each case the curve has a lowest point.

If the coefficient of x^2 is negative, the curve is turned over

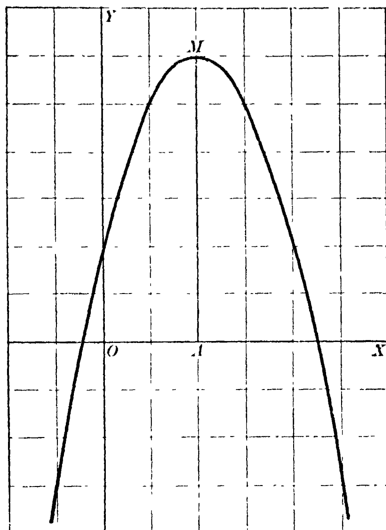


FIG. 11.

and is seen as the curve in Fig. 11, where the graph of the function $-x^2+4x+2$ is shown. In this case the curve has a highest point.

A familiar quadratic function is given by the equation $s=\frac{1}{2}gt^2$ which is the simplest form of the equation of ***accelerated motion.*** The general form of this equation is

$$s=\frac{1}{2}gt^2+v_0t+s_0,$$

where v_0 is the initial velocity and s_0 is the initial space.

EXERCISES

1. Construct the graph of each of the following functions.

(*a*) x^2.	(*d*) $x^2 - 2x$.	(*g*) $x^2 - 2x - 4$.
(*b*) $x^2 + 4$.	(*e*) $x^2 + 2x$.	(*h*) $x^2 + 4x$.
(*c*) $x^2 - 4$.	(*f*) $x^2 - 2x + 4$.	(*i*) $-x^2 - 4x + 3$.

2. Construct the graphs of the functions $x^2 + 4x + 1$, $x^2 + 4x + 2$, $x^2 + 4x + 3$ with the same axes and the same units, and determine as nearly as you can what effect changing the absolute term has upon the graph.

3. Construct with the same axes and the same units the graphs of the functions $x^2 - 2x + 1$, $x^2 - 4x + 1$, and $x^2 - 6x + 1$ and give your opinion as to the influence of the coefficient of x upon the position of the graph.

4. Construct the graph of $x^2/2$ and compare it with graph of x^2 constructed upon the same axes, and with the same units.

5. By means of a table of logarithms construct the graph of the equation

$$y = \log_{10} x.$$

6. By the same means construct the graph of the equation

$$y = 10^x.$$

[Hint. Let the unit for ordinates be 10 times the unit for abscissas.]

52. Maximum and Minimum Values of Quadratic Functions. In § 51 it was stated that the graph of the function $x^2 - 4x + 5$ has a lowest point, and that the graph of the function $-x^2 + 4x + 2$ has a highest point. The number expressing the length of the ordinate drawn to the lowest point of such a graph is called the ***minimum*** value or *least value*, of the function. Similarly, the number expressing the length of the ordinate drawn to the highest point of a graph, like that in Fig. 11, is called the ***maximum*** value or *greatest value* of the function. A minimum value is less, and a maximum value greater, than any other value in the immediate neighborhood.

Maximum and minimum values of quadratic functions are easily found by transforming the function into the sum or the difference of two squares by the method of § 19. For example, the function

$$x^2 - 4x + 5$$

may be written in the form

$$(x - 2)^2 + 1.$$

In this new form the first term $(x - 2)^2$ alone is variable, and, since it is a square, it is either a positive number, or zero, for any real value of x. The *least* value of the function is, therefore, the value that corresponds to the least value of $(x - 2)^2$. Now, the least value of $(x - 2)^2$ is zero. When $(x - 2)^2$ is zero, $x = 2$, and the corresponding value of the function is 1. (See Fig. 10.)

Similarly, the function

$$-x^2 + 4x + 2$$

may be written in the form

$$-(x - 2)^2 + 6.$$

In this case the first term is always negative, or zero, and the *greatest* value of the function corresponds to $(x - 2)^2 = 0$; that is, the maximum value of the function is 6. This value is found by making $x = 2$. (See Fig. 11.)

If the coefficient of x^2 is not unity, the function must first be put in the form

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right);$$

then the factor within the parentheses may be transformed as before. Thus we may write

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right].$$

(See Ex. 2, p. 19.)

EXERCISES

1. Find the maximum, or minimum, as the case may be, of each of the following functions, and give the corresponding value of x.

(a) $x^2 - 2x + 7$.
(b) $x^2 - 3x + 11$.
(c) $-x^2 + 2x + 7$.
(d) $2x^2 - 4x + 9$.
(e) $-2x^2 + 4x + 9$.
(f) $-3x^2 + 2x - 7$.

2. The motion of a body thrown vertically upward in an unresisting medium is given by the equation

$$s = -\tfrac{1}{2}gt^2 + v_0t,$$

where $g = 32.2$, and v_0 is the velocity with which it is thrown. Find the maximum height to which a ball thrown upward with a velocity of 100 feet per second, will rise.

3. Solve the problem in Ex. 2 in general terms, *i.e.* without substituting values for v_0 and g.

4. How long will the ball in Ex. 2 rise if the initial velocity is 100 feet per second?

[HINT. What value of t corresponds to the maximum value of s?]

5. What are the area and the dimensions of the largest rectangle that can be inscribed in a right triangle whose base is 4 feet and altitude 6 feet?

[HINT. Let the area be given by $a = xy$, then by means of similar triangles y can be expressed in terms of x and so a will be expressed as a quadratic function of x.]

6. Find the area and the dimensions of a rectangular ventilating window, of maximum opening, that can be put in a window that is an equilateral triangle whose side is 6 feet.

53. The Power Function. A function of the form

$$ax^n, \tag{11}$$

which is the product of a number a and a power of x, is called a ***power function.*** The great importance of the power function lies in the fact that by means of it many fundamental formulas of geometry and physics may be expressed. The following examples will give some indication of the rôle that it plays.

(a) The circumference of a circle with radius r is the power function $2\pi r$, and the area is the power function πr^2;

(b) the volume of a sphere is the power function $\frac{4}{3}\pi r^3$;

(*c*) the law for a body falling freely in a vacuum is given by the power function $\frac{1}{2}gt^2$;

(*d*) the time required for a body to fall freely through a space s is the power function $\sqrt{2/g}\cdot s^{1/2}$;

(*e*) the Newtonian law of gravitation is given by the equation

$$f=\frac{c}{t^2}=ct^{-2};$$

(*f*) in the so-called adiabatic expansion of air the pressure p expressed in terms of the density ρ is the power function

$$p=c\cdot\rho^{1.408}.$$

The simplest power function is x^n. For $n=1$ the graph is a straight line, as we already know, and for $n=2$ the graph is a parabola whose equation is

$$y=x^2.$$

Figure 12 shows the graphs of $x^{1/2}$, x, x^2, and x^3. The graphs of all power functions of the form x^n, where n is a positive

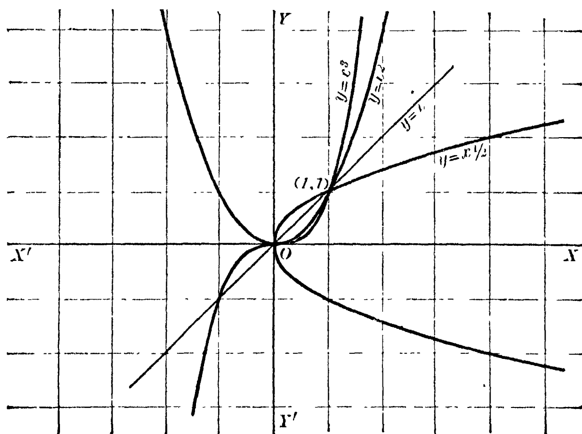


Fig. 12.

integer, or a positive fraction, pass through the two points (0, 0) and (1, 1). Why?

54. Power Functions with Negative Exponents. Infinity. The simplest example of a power function with a negative exponent is x^{-1}, which may be written $1/x$. Brief consideration of this function will serve to bring out a property not found in any of the functions that have been studied heretofore. The graph of the function is the graph of an equation which may be written in any one of the three forms

$$y = x^{-1}, \quad y = \frac{1}{x}, \quad yx = 1.$$

Clearly the graph passes through the point (1, 1), but it does not pass through the point (0, 0), since the substitution of 0 for x would give $y = 1/0$, which has no meaning. (§ 14.) Nevertheless, if we construct a table of values for the function $y = 1/x$ corresponding to the values

$$x = 1, \quad x = .1, \quad x = .01 \cdots$$

we shall see how y behaves as x becomes smaller and smaller.

x	1	.1	.01	.001	.0001	.00001
$y = 1/x$	1	10	100	1000	10,000	100,000

From this table, it is perfectly clear that as x continues to decrease y, or $1/x$, will continue to increase and as x becomes *indefinitely small* y will become *indefinitely large*. We say that y ***becomes infinitely great***, or simply that it ***becomes infinite***, and for the word infinity we use the symbol "∞." The symbol ∞ cannot be looked upon as a symbol for a number like 1, or 2, or 5/6, or 0, or $\sqrt{2}$, since it does not obey the ordinary rules for reckoning by the fundamental operations.

For a negative value of x, the function $y = 1/x$ is negative, taking a value equal in magnitude, but opposite in sign, to the

value for the corresponding positive value of x. Thus for $x = .01$, $y = 100$, and for $x = -.01$, $y = -100$. The graph consists of two *apparently* distinct parts, as in Fig. 13. The curve is called an ***equilateral hyperbola.***

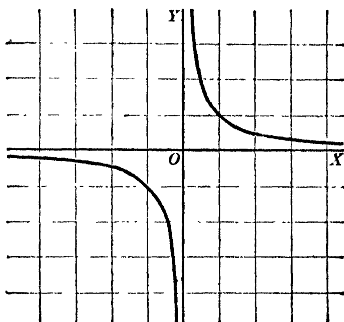


FIG. 13.

The graphs of the functions x^{-n}, where $-n$ is a negative integer, are somewhat similar in shape to the equilateral hyperbola.

EXERCISES

1. Construct the graph of each of the following power functions, using 4 inches, or 10 centimeters, as unit.

(*a*) x^3.	(*e*) $3x^2$.	(*i*) x^{-2}.
(*b*) x^4.	(*f*) $3x^{1/2}$.	(*j*) $x^{-1/3}$.
(*c*) $x^{1/2}$.	(*g*) $x^{2/3}$.	(*k*) $2x^{-2}$.
(*d*) $x^{3/2}$.	(*h*) $2x^{-1}$.	(*l*) $3x^{-1/3}$.

2. Give two points through which all the curves whose equations are of the form

$$y = ax^n,$$

pass, if $n > 0$.

3. Give one point through which all the curves whose equations are of the form

$$y = ax^{-n},$$

pass, if $n > 0$.

55. Variation. We say that one number y *varies as*, or is *proportional to*, another number x, if y is equal to a constant times x, that is if

$$y = cx.$$

If one number y is related to another number x according to the relation

$$y = cx^{-1},$$

we say y varies *inversely* as x.

In general, if

$$y = cx^n, \tag{12}$$

where n is any rational number, positive or negative, integral or fractional, we say that y varies as the nth power of x.

The connection between variation and proportion is easily established. For, if x_1 and x_2 be two values of x, and y_1 and y_2 the corresponding values of y, according to the relation (12), then $y_1 = cx_1^n$ and $y_2 = cx_2^n$. When one of these equations is divided by the other the resulting equation may be written as the proportion

$$y_1 : y_2 = x_1^n : x_2^n. \tag{12 a}$$

Conversely, if equation (12 a) holds for every pair of values of x and y, then

$$y_1 = \frac{y_2}{x_2^n} x_1^n.$$

But if x_2 is fixed, y_2 is fixed, and y_2/x_2^n is a constant which may be denoted by c. Then $y_1 = cx_1^n$. The last equation is identical in form with equation (12). Consequently, the equation (12) is identical in meaning with the proportion (12 a).

In practical problems the constant c is usually determined by experiment. For example, if a body travels with uniform velocity, the distance traversed is proportional to the time, and we have $s = ct$. If we measure the distance s corresponding to a definite time t, c is known immediately. It is, of course, the velocity.

Again, under the Newtonian law of gravitation, the force by which two unit masses are attracted to each other varies inversely as the distance between them. The formula for the force is therefore

$$f = c\frac{1}{d^2}.$$

To find c experimentally, it would be necessary to take two bodies of unit mass, measure the distance between them, and then measure the force f.

The determination of the constant c in variation problems depends upon the units employed in measuring the numbers that are proportional to each other. In the well-known formula for the motion of a body falling from rest, $s = \frac{1}{2}gt^2$, the constant g is approximately 32.2 only when t is measured in seconds and y in feet; if t is measured in seconds and s in centimeters, g is approximately 981.

When in equations like $s = ct^2$, t denotes time and s space, the number $2c$ is called the ***acceleration.***

EXERCISES

1. The weight w that can be raised by a pull of p lb. by means of a pulley block varies as p, that is, $w = kp$. Find k if $w = 250$ when $p = 100$.

2. The height h of an object varies as the length l of its shadow at any given place and time. If the shadow of a pole 10 feet high is 8 feet long, find the relation between h and l. Hence find the height of a building whose shadow is 65 feet long.

3. The amount of extension e of an iron rod under tension of p lb. varies as p. For a certain rod $e = 0.009$ in. when $p = 1200$ lb. Express e in terms of p.

4. The space traversed by a ball rolling down a rough inclined plane varies as the square of the time. If the ball rolls 23 feet in 2 seconds, what is the acceleration?

5. There are 2.54 centimeters in 1 inch. If in the formula $s = \frac{1}{2}gt^2$ s is measured in centimeters and t in seconds, what is the acceleration?

6. The ordinate of a moving point varies as the square of the abscissa, and when the abscissa is 4 the ordinate is 64. What is the equation between x and y?

7. We know that the circumference of a circle varies as the radius, *i.e.* if c and r denote circumference and radius, respectively, $c = kr$. Measure the circumference of a circle with known radius and thus determine k.

8. Boyle's law states that the volume v of a given quantity of gas, at constant temperature, varies inversely as the pressure p, measured in pounds per square inch. If, for a certain gas $v = 2$ cu. ft. when $p = 91$ lb., find the relation between p and v. Hence find v when $p = 200$ lb.

56. Discontinuous Functions. In the previous section graphs were discovered which contained a break or gap. For example, in the graph of x^{-1} it would not be possible to trace out with a pencil point the curve between $x = -1$ and $x = +1$ without removing the pencil from the paper. We say that the function is ***discontinuous*** at $x = 0$.

Other examples of discontinuous functions are numerous. For example, the postage on written matter is "2 cents per ounce or each fraction." The postage to be paid on x ounces is $2x$ cents provided x is an integer. The dotted straight line of Fig. 14 does not, however, represent the situation, for the postage is the same for all weights between zero and one ounce, then between one ounce and two ounces, and so on. The real situation is indicated by the series of unit segments of straight lines all parallel to the x-axis, and each having its right-hand extremity in the line $y = 2x$, as in Fig. 14. The function is discontinuous at $x = 0, x = 1, x = 2, \cdots$, constant for fractional, and nonexistent for negative values of x.

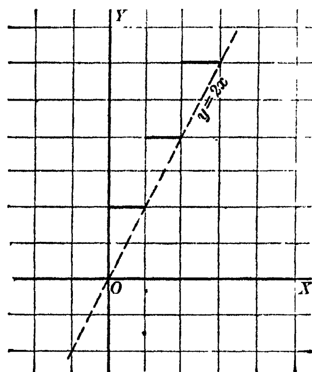


FIG. 14.

57. Statistical Graphs. One of the most common, and at the same time most useful, applications of graphical methods is the representation of tables of statistics by graphs. Such a graph is called a ***statistical graph.*** Statistical graphs are employed extensively in the study of political economy and kindred sub-

jects, and, indeed, in the study of all subjects which involve the collection and comparison of numerous data.

For example, suppose it be required to represent graphically the data of the following table giving the population of the United States for each decade from 1850 to 1910 inclusive.

YEAR	POPULATION
1850	23,000,000
1860	31,000,000
1870	39,000,000
1880	50,000,000
1890	63,000,000
1900	76,000,000
1910	92,000,000

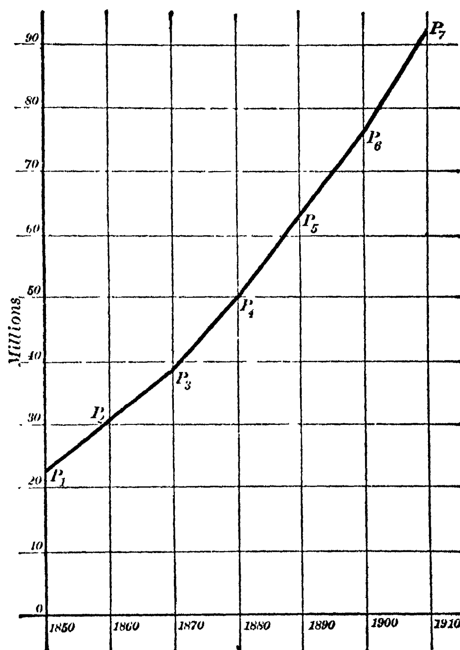


FIG. 15.

Taking 10,000,000 as the unit of measurement on the population-axis and 10 years as the unit on the time-axis, it is not difficult to see that the known points of the graph are the points P_1, P_2, P_3, P_4, P_5, P_6, and P_7, of Fig. 15.

It is usual to assume that for the first approximation the change is uniform between two consecutive known points so that the portion of the graph between two such points is a segment of a straight line. The graph is then a broken line which may change its direction abruptly from time to time. If desired the graph may be "smoothed out" by drawing a smooth curve through the known points.

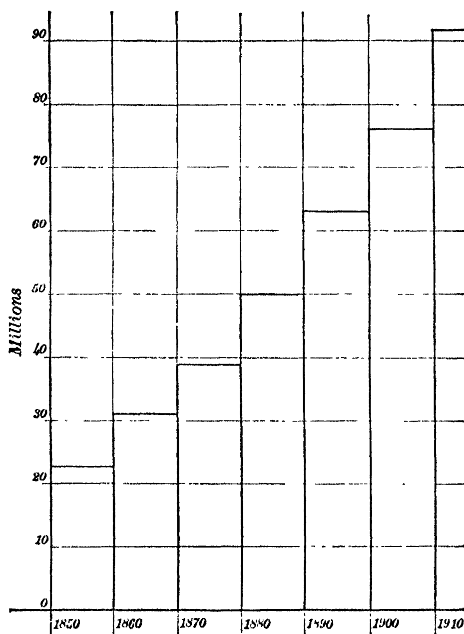


FIG. 16.

Another common form of the statistical graph is obtained by drawing a rectangle with a unit base and altitude equal to the length of the ordinate. The value of the function corresponding to a given value of the independent variable is then represented by an area instead of by a length.

In the example just given the graph would be obtained by constructing a series of rectangles each with unit base and having for their altitudes the ordinates of lengths 2.3, 3.1, 3.9, and so on. The graph is shown in Fig. 16.

Another form of statistical graph is obtained by using the number of individuals reaching a certain standard as the ordinate and the standard itself as abscissa. Suppose a class is marked according to the following table:

Number of Students	10	20	35	50	75	78	50	20
Grade	50	60	70	75	80	85	90	95

The graph obtained by locating the points (10, 50), (20, 60), (35, 70), and so on is called the ***curve of distribution*** of the marks.

It is important to note that for a statistical graph, no formula for the function need be known. Indeed in most cases occurring in statistics, no function formula can be known.

EXERCISES

1. What does the graph in Fig. 15 show about the period of most rapid growth in population?

2. The high school attendance in the United States since 1876 is given by the following table:

Year	1876	1880	1890	1900	1910
Attendance . . .	22,986	26,609	202,963	519,251	915,061

Compare graphically the increase in high school attendance with the increase in population, taking the data for population from the table given above. Use both forms of graph.

3. The following table gives the average weights of men of different heights and ages. Plot graphs of each of the rows and compare them.

Height	Age 15–24	Age 25–29	Age 30–34	Age 35–39	Age 40–44	Age 45–49	Age 50–54	Age 55–59	Age 60–64	Age 65–69
	Lb.	Lb	Lb.	Lb	Lb.	Lb.	Lb.	Lb.	Lb.	Lb.
5 feet	120	125	128	131	133	134	134	134	131	—
5 feet 1 inch . . .	122	126	129	131	134	136	136	136	134	—
5 feet 2 inches . .	124	128	131	133	136	138	138	138	137	—
5 feet 3 inches . .	127	131	134	136	139	141	141	141	140	140
5 feet 4 inches . .	131	135	138	140	143	144	145	145	144	143
5 feet 5 inches . .	134	138	141	143	146	147	149	149	148	147
5 feet 6 inches . .	138	142	145	147	150	151	153	153	153	151
5 feet 7 inches . .	142	147	150	152	155	156	158	158	158	156
5 feet 8 inches . .	146	151	154	157	160	161	163	163	163	162
5 feet 9 inches . .	150	155	159	162	165	166	167	168	168	168
5 feet 10 inches . .	154	159	164	167	170	171	172	173	174	174
5 feet 11 inches . .	159	164	169	173	175	177	177	178	180	180
6 feet	165	170	175	179	180	183	182	183	185	185
6 feet 1 inch . . .	170	177	181	185	186	189	188	189	189	189

4. The following table gives certain statistics of the Post Office during the years 1902–1915. Draw graphs of all the columns and compare them

UNITED STATES POST-OFFICE STATISTICS

Fiscal Years	Number of Post Offices	Extent of Post Routes in Miles	Revenue of the Department	Expenditure of the Department	Amount Paid for	
					Compensation to Postmasters	Transportation of the Mail
1907	62,659	463,406	$183,585,005	$190,238,288	$24,575,696	$81,090,849
1908	61,158	450,738	191,478,663	208,351,886	25,599,397	81,381,421
1909	60,144	448,618	203,562,383	221,004,102	26,569,892	84,052,596
1910	59,580	447,998	224,128,657	229,977,224	27,521,013	85,259,102
1911	59,237	435,388	237,879,823	237,648,926	28,284,964	88,058,922
1912	58,729	436,469	246,744,015	248,525,450	28,467,726	89,154,811
1913	58,020	436,293	266,619,525	262,067,541	29,126,662	92,278,517
1914	56,810	435,597	287,934,565	283,543,769	29,968,515	98,002,421
1915	56,380	433,334	287,248,165	298,546,026	30,400,145	104,701,200

5. Plot a graph for the following table of velocities of water with a given "head."

THEORETICAL VELOCITY OF WATER IN FEET PER SECOND

Head, Feet	Velocity, Feet per Second	Head, Feet	Velocity, Feet per Second	Head, Feet	Velocity, Feet per Second	Head, Feet	Velocity, Feet per Second
10	25.4	25	40.1	55	59.5	85	74.0
12	27.8	30	43.9	60	62.1	90	76.1
15	31.1	35	47.4	65	64.7	95	78.2
18	34.0	40	50.7	70	67.1	100	80.3
20	35.9	45	53.8	75	69.5	125	89.7
22	37.6	50	56.7	80	71.8	150	98.3

58. Interpolation of Values of Functions. The construction of a graph is usually accomplished by finding the values of the function for a few isolated values of the independent variable, and then drawing a smooth curve through the ex-

tremities of the ordinates so determined. Frequently the values of the function for intermediate values of the independent variable are not less important than the values for which the independent variable is given. The process of finding these intermediate values without reference to the function itself is called ***interpolation.*** We have already made use of this process in connection with the use of logarithmic tables.

If the graph of the function whose values are to be interpolated is a straight line the intermediate values may be

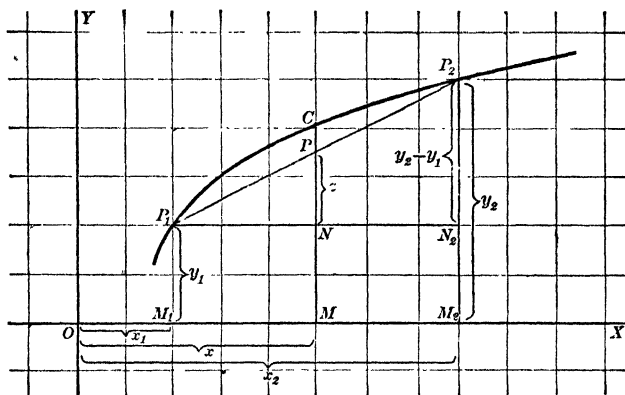


Fig. 17.

found exactly. To find the correction to be added to y suppose y_1 and y_2 are known values of the function corresponding to the values x_1 and x_2 of the independent variable.

Let $OM_1 = x_1$, $OM_2 = x_2$, $M_1P_1 = y_1$, $M_2P_2 = y_2$, and let $MP = y$ be the unknown value to be interpolated between y_1 and y_2. (See Fig. 17.) Draw P_1N_2 parallel to the x-axis. Then $z = NP$ is the correction that must be added to y_1 in order to obtain $y = MN + NP$. For, from the similar triangles NP_1P and $N_2P_1P_2$,

$$NP : N_2P_2 :: P_1N : P_1N_2.$$

But $NP = z$, $N_2P_2 = y_2 - y_1$, $P_1N = x - x_1$, and $P_1N_2 = x_2 - x_1$, so that the proportion may be written

$$z : y_2 - y_1 = x - x_1 : x_2 - x_1.$$

Consequently,

$$z = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1). \tag{13}$$

In this expression $x - x_1$, $x_2 - x_1$, and $y_2 - y_1$ are all known, so that the correction z is readily found. If the function increases with x the correction must be added to y_1, if it decreases with x the correction must be subtracted from y_1.

If the graph of the function is not the straight line P_1P_2 but the curve P_1CP_2, the true correction would be NC and not NP. However, for elementary work it is sufficiently accurate to assume that the graph is a straight line between two near-by points P_1 and P_2. The correction is therefore found in all cases by means of formula (13).

Example. Suppose we have given

$$(5.56)^2 = 30.9136 \quad \text{and} \quad (5.57)^2 = 31.0249$$

and wish to find $(5.5642)^2$ to the same number of decimal places. The problem is therefore to interpolate a value of the function x^2 between the values for 5.56 and for 5.57. The difference $y_2 - y_1$ is .1113. Also we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{.0042}{.0100} = \frac{42}{100}.$$

The correction z is, therefore,

$$\frac{42}{100} \times .1113 = .0467,$$

and we have approximately,

$$(5.5642)^2 = 30.9603.$$

This result could have been found easily by other means, but there are many cases where interpolation is practically the only method available.

EXERCISES

1. $\sqrt{556} = 23.5797$, and $\sqrt{557} = 23.6008$. Find $\sqrt{556.34}$.

2. $\sqrt[3]{460} = 7.7194$, and $\sqrt[3]{461} = 7.7250$. Find $\sqrt[3]{460.56}$.

3. The areas of two circles with radii 6.52 and 6.53, are 33.3878 and 33.4901, respectively. Find the area of a circle with radius 6.5231.

4. Log 2513 = 3.40019 and log 2514 = 3.40037. Find log 2513.7.

5. The population of Chicago was 1,698,575 in 1900 and 2,185,283 in 1910. What was it in 1906?

59. Zeros and Infinities of a Function. A ***zero*** of a given function is a value of the variable x for which the value of the function is zero. In symbols x_1 is a zero of $f(x)$ if $f(x_1) = 0$. An ***infinity*** of a function is a value of x for which the function becomes infinite. Thus, 2 is a zero of the function $2x - 4$. Again, 2 is an infinity of the function $1/(x-2)$, or of the function $(x^2 - 7x + 11)/(x - 2)$. The real zeros of a function are represented graphically by the abscissas of points where the graph crosses the x-axis.

EXERCISES

1. Point out zeros and infinities of the following functions, as far as possible.

(*a*) $2x + 4$.
(*b*) $\frac{1}{3}x + 1$.
(*c*) $\frac{1}{2}x - 2$.
(*d*) $2x$.
(*e*) $\sqrt{x-3}$.
(*f*) $\sqrt{4-x}$.
(*g*) $(x-1)(x-2)$.
(*h*) $(x+3)(2x+1)$.
(*i*) $(5-x)(3-x)$.
(*j*) $\dfrac{1}{x-2}$.
(*k*) $\dfrac{3}{\sqrt{4-x^2}}$.
(*l*) $\dfrac{a}{\sqrt{a^2-x^2}}$.
(*m*) $\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$.
(*n*) $\dfrac{x+4}{x\sqrt{9-x^2}}$.
(*o*) $\dfrac{x-2}{\sqrt{x^2-4}}$.

2. The graph of a certain continuous function $f(x)$ is negative for $x = a$ and positive for $x = b$. Show by graphical representation that the function has at least one zero between $x = a$ and $x = b$. Can it have an even number of distinct zeros between $x = a$ and $x = b$?

60. Implicit Functions. If an equation exists between two variables, it is frequently possible to solve the equation for one variable in terms of the other. The first variable is then expressed as a function of the second. For example, if

$$x^2 + y^2 = 4,$$

we may transpose the term x^2 and take the square root of both sides of the resulting equation. The final result is given by the equation

$$y = \sqrt{4 - x^2},$$

which is implied in the equation

$$x^2 + y^2 = 4.$$

A function which is implied by a mathematical relation in the form of an equation is called an ***implicit function***. The function exists even though it may be impossible to find its form, as in the example just given.

The graph of an implicit function may always be constructed if the form of the function can be found; otherwise the construction is usually impossible by elementary means.

As in the example given above, the process of finding the form of the function frequently leads to a function having more than one value for each value of x. The function $y = \sqrt{4 - x^2}$ found from the equation $x^2 + y^2 = 4$, has two values for every value of x, one positive, and the other negative.

EXERCISES

1. Construct the graph of the function of x determined by each of the following equations.

(*a*) $x^2 + y^2 = 4$.
(*b*) $x^2 - y^2 = 4$.
(*c*) $2x + 3y + 7 = 0$.
(*d*) $xy = 2$.
(*e*) $x^{-1}y = 2$.
(*f*) $x^3y = 7$.

CHAPTER VI

QUADRATIC EQUATIONS WITH ONE UNKNOWN

61. Definitions. A quadratic equation with one unknown is an equation which has, or may be reduced to, the standard form

$$(1) \qquad ax^2 + bx + c = 0,$$

where a, b, and c are constants, and x is the unknown number. The equations

$$3x^2 - 5x + 4 = 0, \quad 5x^2 = 6 - 3x, \quad 3x = \frac{5x - 1}{3x},$$

are all quadratic.

The numbers a, b, and c are called the ***coefficients*** of the equation. The number c is called the ***constant*** or the ***absolute term.***

A number is said to ***satisfy*** an equation if, when it is substituted for the unknown number, the two members of the equation become numerically, or identically, equal.

For example, 2 satisfies the equation

$$x^2 - 5x + 6 = 0,$$

since

$$2^2 - 5 \cdot 2 + 6$$

is numerically equal to zero.

Again, b/a satisfies the linear equation

$$ax = b,$$

since

$$a \times (b/a) \equiv b.$$

A ***root*** of an equation with one unknown is a number which satisfies the equation.

62. Solution of the Quadratic Equation. If the equation

(1) $$ax^2 + bx + c = 0$$

be divided through by a, and the term c/a in the resulting equation be transposed, it is reduced to the form

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square of the left member by adding $b^2/4\,a^2$ to both sides, we obtain the equation

$$x^2 + \frac{b}{a}x + \frac{b^2}{4\,a^2} = \frac{b^2}{4\,a^2} - \frac{c}{a};$$

or,

$$\left(x + \frac{b}{2\,a}\right)^2 = \frac{b^2 - 4\,ac}{4\,a^2}.$$

Extracting the square root of both sides of this equation, we have

$$x + \frac{b}{2\,a} = \pm\frac{\sqrt{b^2 - 4\,ac}}{2\,a},$$

and finally, by transposing the known term $\frac{b}{2\,a}$,

(2) $$\boldsymbol{x = -\frac{b}{2\,a} \pm \frac{\sqrt{b^2 - 4\,ac}}{2\,a}}.$$

Equation (2) gives two values for x in terms of the coefficients of equation (1), one for the positive and the other for the negative sign of the radical part. That both these values satisfy equation (1) and are, therefore, roots of it, may be seen by actually substituting the values for x and showing that the resulting equality is an identity.

Equation (2) may be looked upon as a formula for the solution of *any* quadratic equation which has been reduced to the form (1). It is much better, however, to master the process and to solve each quadratic equation by the method used above.

Thus, to solve the equation

$$3x^2 - 5x - 6 = 0,$$

it is better to go through the successive steps :

$$x^2 - \tfrac{5}{3}x - 2 = 0,$$

$$x^2 - \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{25}{36} + 2 = \tfrac{97}{36},$$

$$x - \frac{5}{6} = \pm\frac{\sqrt{97}}{6},$$

$$x = \frac{5}{6} \pm \frac{\sqrt{97}}{6}.$$

EXERCISES

1. Solve each of the following equations.

(*a*) $x^2 - 7x - 11 = 0.$

(*b*) $2x^2 - 7x - 13 = 0.$

(*c*) $2x = 12 - 11x^2.$

(*d*) $3x - 7 = 4x^2.$

(*e*) $ax^2 - 14 = bx.$

(*f*) $mx^2 - nx = p.$

(*g*) $x^4 - 5x^2 + 6 = 0.$

(*h*) $x^6 - 5x^3 + 6 = 0.$

(*i*) $(x - a)^2 + 3(x - a) - 14 = 0.$

(*j*) $6(m + n)^2 + 5(m + n) - 4 = 0.$

2. Find t from the equation

$$s = \tfrac{1}{2}gt^2 + v_1t + v_2.$$

3. The difference between two numbers is 4 and their product is 21. What are the numbers? Explain the double result.

63. Solution by Factoring. By the method of § 19 every quadratic expression may be reduced to the difference of two squares. For, clearly,

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

If $b^2/4a^2$ be added to and subtracted from the expression in the parenthesis, the quadratic equation takes the form

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2}\right) = 0,$$

or,

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2\right] = 0.$$

By the method of factoring the difference of two squares, the equation is reduced to the form

$$(3)\quad a\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)=0.$$

By hypothesis, a is not zero; consequently, either

$$(4)\quad x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}=0,\quad \text{or}\quad x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}=0,$$

since a product cannot be zero unless at least one of its factors is zero. The two equations (4) are both linear. If r_1 and r_2 denote their roots, then

$$(5)\quad r_1=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a},\qquad r_2=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

These solutions are identical with (2) of § 62.

If $\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$ and $\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ in equation (3) be replaced by their values $-r_1$ and $-r_2$, we may write

$$(6)\qquad \boldsymbol{ax^2+bx+c=a(x-r_1)(x-r_2)},$$

where r_1 and r_2 are the roots of the quadratic equation.

From the identity (6) it follows that a quadratic equation may be solved by factoring the left member, *after the equation has been reduced to the standard form.*

EXERCISES

1. Solve each of the following equations by factoring.

(a) $x^2-5x+6=0.$

(b) $x^2-7x+10=0.$

(c) $x^2-5x+7=0.$

(d) $9x^2-12x+4=0.$

2. Solve each of the following equations by factoring. First reduce the right-hand side to zero.

(*a*) $x^2 = 10x + 21$. (*c*) $(x-3)(x-4) = \frac{1}{4}$.
(*b*) $12x^2 = 17x - 6$. (*d*) $(x-2)(x-5) = 4$.

3. Solve the equation $x^3 = 64$.

[Hint. Transpose the constant term and factor.]

4. Solve the equation $x^4 = 81$.

5. By means of the identity

$$ax^2 + bx + c \equiv ax^2 + bx + c - (ar^2 + br + c),$$

where r is a root of the equation $ax^2 + bx + c = 0$, prove that $x - r$ is a factor of the quadratic function $ax^2 + bx + c$.

64. The Formation of a Quadratic Equation having Known Roots. By means of the identity (6), it is possible to build up a quadratic equation having any two numbers, real or imaginary, as its roots.

If, for example, the roots of a quadratic equation are 3 and 5, the equation must be

$$(x-3)(x-5) = 0, \quad \text{or} \quad x^2 - 8x + 15 = 0.$$

EXERCISES

1. Construct the equations whose roots are as follows.

(*a*) 2, 4. (*d*) $2+\sqrt{3}$, $2-\sqrt{3}$. (*g*) 1, $\sqrt{3}$.
(*b*) 3, -7. (*e*) $p+\sqrt{q}$, $p-\sqrt{q}$. (*h*) $-\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}$, $-\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}$.
(*c*) -4, -3. (*f*) $-\sqrt{3}$, $\sqrt{3}$. (*i*) 0, 2.

2. What is the simplest quadratic equation whose roots are both zero?

3. Prove that if the roots of a quadratic equation are conjugate radicals $A+\sqrt{B}$ and $A-\sqrt{B}$, where A and B are rational integers, the coefficients of the quadratic are rational integers. Is the converse true?

4. One root of a quadratic equation with rational coefficients is $3+\sqrt{5}$. What is the equation?

5. Prove that the expression $ax^2 + bxy + cy^2$ may be resolved into two linear factors that are rational in x and y, though not necessarily rational in the coefficients of x and y.

6. In view of what has gone before, can you factor $x^2 + y^2$?

65. Character of the Roots. The roots of a quadratic equation

$$ax^2 + bx + c = 0,$$

may be either real or imaginary. (See § 33.) If a, b, and c are real numbers, the sign of the quantity $b^2 - 4ac$ determines the character in this respect of the two roots

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Clearly, if $b^2 - 4ac$ is *positive*, the roots are both *real*; and if $b^2 - 4ac$ is *negative*, the roots are both *imaginary*. Moreover, if $b^2 - 4ac$ is *zero*, the roots both reduce to $-b/2a$, and are therefore *equal*. These results may be stated in the following theorem:

The roots of the quadratic equation

$$ax^2 + bx + c = 0,$$

with real coefficients, are

(*a*) *real and unequal, if* $b^2 - 4ac > 0$;

(*b*) *real and equal, if* $b^2 - 4ac = 0$;

(*c*) *imaginary, if* $b^2 - 4ac < 0$.

To say that the roots of a quadratic equation are imaginary does not mean that they do not exist, but rather that the facts given by the equation are incapable of geometrical or physical interpretation in the ordinary sense. For example, if the hypotenuse of a right-angled triangle be 20 and one leg 25, the third side is given by the equation

$$x^2 + 25^2 = 20^2,$$

or,

$$x^2 + 225 = 0.$$

The roots of this equation are imaginary and, as we know, the triangle is impossible.

Another characterization of the roots is fully as important as the preceding one.

The roots of a quadratic equation with rational coefficients are

(*a*) *rational, if* $b^2 - 4\,ac$ *is zero or is a perfect square ;*

(*b*) *irrational, if* $b^2 - 4\,ac$ *is neither zero nor a perfect square.*

The number $b^2 - 4\,ac$ is called the ***discriminant*** of the quadratic equation.

EXERCISES

1. Find the discriminant of each of the following equations.

(*a*) $x^2 - 5\,x + 6 = 0.$ (*c*) $4\,x^2 = 12\,x + 9.$

(*b*) $3\,x^2 - 7\,x + 8 = 0.$ (*d*) $5\,x = \dfrac{4 - x}{x}.$

2. Determine the character of the roots in each of the following equations without solving.

(*a*) $x^2 - 10\,x + 40 = 0.$ (*b*) $3\,x^2 + 14\,x - 16 = 0.$ (*c*) $10\,x^2 - 17\,x + 1000 = 0.$

3. What value of k will make the roots of the equation

$$3\,x^2 - 6\,x + 2\,k = 0$$

equal ?

4. Suppose the equation $5\,x^2 + bx + 20 = 0$ is subjected to the condition that its roots must be equal. What is the value of b ?

5. For what values of b will the equation in problem 4 have imaginary roots ?

6. The law of motion for a body thrown vertically upward is given by the equation $s = -\,16.1\,t^2 + vt$ where v is the velocity with which it is thrown, s the space, and t the time. If $v = 100$ feet per second, in what time will the body be at a distance of 100 feet above the earth ? Interpret the results.

7. Under the conditions of problem 6, at what time will it be at a height of 25,000/161 feet ?

8. Under the conditions of problem 6, at what time will the body be at an altitude of 200 feet ? Interpret the result.

9. What must be the initial velocity in order that a body thrown vertically upward shall rise exactly 100 feet ?

10. Prove that all quadratic equations for which the signs of the coefficient of x^2 and the absolute term are unlike have real roots.

66. Geometric Interpretation. From the definition of the roots of a quadratic equation

$$ax^2 + bx + c = 0,$$

it is clear that they are the zeros (§ 59) of the quadratic function

$$ax^2 + bx + c.$$

The roots are, therefore, represented graphically by the distances from the origin to the points where the graph of the

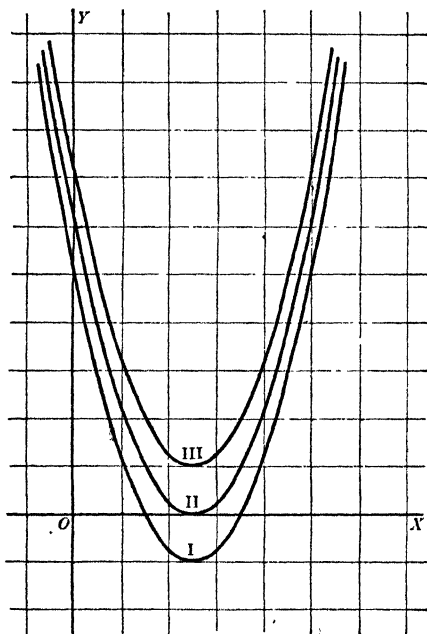


FIG. 18.

function $ax^2 + bx + c$ crosses the x-axis. If the discriminant $b^2 - 4ac$ of the quadratic equation is *positive*, the graph crosses the x-axis in *two distinct points* as in I (Fig. 18). If $b^2 - 4ac$

is equal to *zero*, it *touches* the x-axis, which is sometimes expressed by saying that the graph crosses the axis in *two coincident points*. If $b^2 - 4ac$ is *negative*, the graph *does not cut* the x-axis, but lies wholly on one side of it. In the last case we may say that the graph cuts the x-axis in *two imaginary points*.

It follows, therefore, that when $b^2 - 4ac \leqq 0$, the sign of $ax^2 + bx + c$ is the same for every value of x except when the function is zero. If $b^2 - 4ac > 0$, however, $ax^2 + bx + c$ is positive for some values of x and negative for others. For this reason the function $ax^2 + bx + c$ is said to be *definite* when $b^2 - 4ac \leqq 0$ and *indefinite* when $b^2 - 4ac > 0$.

NOTE. If the quadratic expression is definite, a single value of x will suffice to determine whether the sign is always positive or always negative.

EXERCISES

1. The graphs I, II, and III in Fig. 17 are the graphs of the functions $x^2 - 5x + 7.25$, $x^2 - 5x + 5.25$, and $x^2 - 5x + 6.25$. To which function does each graph belong? Prove your statements.

2. Tell without constructing the graphs of the following functions which graphs cross the x-axis, which touch it, and which do not cross it:

(*a*) $3x^2 + 7x - 6$. (*b*) $3x^2 - 7x + 6$. (*c*) $9x^2 - 24x + 16$.

3. Sketch roughly the graph of a quadratic function belonging to the quadratic equation whose roots are 4 and 5. Is there more than one such graph?

67. The Sum and the Product of the Roots. When the right member of the identity (6) of § 63 is multiplied out it takes the form

$$ax^2 + bx + c \equiv ax^2 - a(r_1 + r_2)x + ar_1r_2. \tag{7}$$

Then

$$bx + c \equiv -a(r_1 + r_2)x + ar_1r_2. \tag{8}$$

Since an identity is true for every value of x for which both members are defined (§ 16), the value $x = 0$ gives

$$ar_1r_2 = c,$$

or,

$$r_1r_2 = \frac{c}{a}. \tag{9}$$

Also on account of (9), the identity (7) is further reduced to the form

$$bx \equiv -a(r_1 + r_2)x,$$

from which it follows that

$$r_1 + r_2 = -\frac{b}{a}. \tag{10}$$

EXERCISES

1. Write down the sum and the product of the roots of each of the following equations without solving the equation.

(*a*) $3x^2 - 5x + 16 = 0$. (*c*) $8x^2 - 17 = 4x$.

(*b*) $-x^2 + 7x + 6 = 0$. (*d*) $x + \frac{8x + 4}{x} = 0$.

2. Find $r_1 + r_2$ from formulas (5), § 63, by direct addition.

3. Find r_1r_2 from formulas (5), § 63, by direct multiplication.

4. Find the value of the symmetric function $r_1^2 + r_2^2$ of the roots of the equation $ax^2 + bx + c = 0$ in terms of the coefficients.

[HINT. Square both sides of the relation $r_1 + r_2 = -b/a$.]

5. Find the value of $r_1 - r_2$ for the equation $ax^2 + bx + c = 0$ without using formulas for r_1 and r_2.

[HINT. From $(r_1 + r_2)^2$ subtract $4r_1r_2$.]

6. If x_1 and x_2 are roots of $ax^2 + bx + c = 0$, find the equation whose roots are x_1^2 and x_2^2.

[HINT. What are the coefficient of x and the absolute term in the new equation?]

7. If x_1 and x_2 are roots of $ax^2 + bx + c = 0$, find the equation whose roots are x_1^2/x_2 and x_2^2/x_1.

68. Equations containing Radicals. The unknown number in an equation frequently occurs under one or more radical signs. After clearing the equation of fractions both sides must be raised to powers corresponding to the indices of the radical signs as often as need be to remove the radicals.

For example, the equation

$$\sqrt{6x - 11} = x - 3$$

must be squared once. The resulting equation is easily reduced to the equivalent equation

$$x^2 - 12x + 20 = 0,$$

whose roots are $x_1 = 2$ and $x_2 = 10$.

In solving such equations great care must be exercised since raising both sides of an equation is liable to introduce ***extraneous roots***, that is to say, roots which do not satisfy the original equation. Any such extraneous root must be discarded, as it is not, properly speaking, a root.

For example, the equation

$$x - 2 = 3$$

has a single root, $x = 5$. But if both sides are squared, the resulting equation is a quadratic equation

$$x^2 - 4x + 4 = 9,$$

which has the two roots, $x = 5$, and $x = -1$. The root $x = -1$ does not satisfy the original equation and is therefore extraneous.

If it be stipulated that in the equation

$$\sqrt{6x - 11} = x - 3$$

the radical shall have the positive sign only, the root $x = 2$ obtained above will not satisfy it, while if it be stipulated that the radical shall have the negative sign, the root $x = 10$ is extraneous.

In solving equations containing radicals it is therefore necessary *to test every root found by substituting it in the original equation.* Unless stated explicitly to the contrary, it is under-

stood that every radical expression in the given equation is to be taken with the positive sign.

EXERCISES

1. Find the values of the unknowns in each of the following equations.

(*a*) $\sqrt{x+9}-\sqrt{x}=1.$

(*b*) $\sqrt{x+7}+\sqrt{x}=7.$

(*c*) $x+\sqrt{c^2-ax}=\dfrac{c^2}{\sqrt{c^2-ax}}.$

(*d*) $x^2+11+\sqrt{x^2+11}=42.$

(*e*) $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}.$

(*f*) $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$

(*g*) $\sqrt{a^2-x}+\sqrt{b^2+x}=a+b.$

(*h*) $\dfrac{\sqrt{x}+\sqrt{x-5}}{\sqrt{x}-\sqrt{x-5}}=\dfrac{2x-5}{5}.$

[HINT. Rationalize the denominator.]

(*i*) $\dfrac{6\sqrt{x}+\sqrt{x-6}}{\sqrt{x}-\sqrt{x-6}}=2x-6.$

2. Solve the equation $+\sqrt{x}+1=0$ and verify the solution. What is the cause of the difficulty?

CHAPTER VII

SYSTEMS OF LINEAR EQUATIONS. DETERMINANTS

69. Two Linear Equations with Two Unknowns. In many problems it is required to find several unknown numbers which satisfy several equations at the same time. Such a set of equations is called a ***system of simultaneous equations***, or, briefly, a ***system of equations.*** The simplest system of equations consists of two ***linear*** equations, that is, two equations of first degree with two unknowns. Such a system may be written in the form

$$(1)\qquad \begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$$

By the well-known method of multiplying the first of these equations by b_2, and the second by b_1, then subtracting the two equations thus found, it may be shown that

$$(2)\qquad x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}.$$

Similarly, using a_2 and a_1 as multipliers, we find

$$(3)\qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

If each of the expressions on the right of (2) and (3) has a meaning, that is, if the common denominator is not zero, the pair of values of x and y is called a solution of the system, according to the following definition:

A solution of a system of equations is a set of values, one for each unknown, which satisfies every equation of the system.

The final test of a solution is that when the values of the set are substituted for the unknowns, all the equations of the system reduce to numerical equalities or to identities.

EXERCISES

1. Solve each of the following systems of equations and verify the solutions found.

(*a*) $\begin{cases} 3x + 7y = 10, \\ 8x - 4y = 20. \end{cases}$

(*b*) $\begin{cases} 7x - 4y - 14 = 0, \\ 8x + 3y - 12 = 0. \end{cases}$

(*c*) $\begin{cases} 3 = \dfrac{4x}{y} + \dfrac{10}{3}, \\ 8 - 4y = 3x. \end{cases}$

(*d*) $\begin{cases} 5x = 3y + 14, \\ 8 = 2x. \end{cases}$

70. Determinants. The expression $a_1b_2 - a_2b_1$, which occurs in the denominators of both x and y, is called a ***determinant of second order***, and when it has reference to the system of equations (1), it is called the ***determinant of the system.*** The determinant $a_1b_2 - a_2b_1$ is usually written as a square array in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4}$$

The numbers a_1, a_2, b_1, b_2, are called the ***elements*** of the determinant. The terms a_1b_2 and a_2b_1 are formed by taking one element, and only one, from each row and each column of the square array.

Clearly, $c_1b_2 - c_2b_1$ and $a_1c_2 - a_2c_1$ are also determinants which may be obtained from the determinant $a_1b_2 - a_2b_1$ by replacing, first, the a's by the c's and, second, the b's by the c's. Hence the solution of the system (1), § 69, may be written in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{5}$$

The equations (5) may therefore be translated into the following rule, called ***Cramer's rule***, for writing down mechanically the solution of any system of linear equations in which the number of equations is equal to the number of unknowns.

For the denominators, write the determinant of the system. For the numerator of each unknown, write the determinant obtained from the determinant of the system by replacing the coefficients of the unknown by the absolute terms.

EXERCISES

1. Solve the following simultaneous systems.

(a) $\begin{cases} 2x + 3y = 6, \\ 3x - 5y = 7. \end{cases}$

(b) $\begin{cases} 7x - 3y = 12, \\ 6x + 4y = 17. \end{cases}$

(c) $\begin{cases} 5x - 4y + 14 = 0, \\ 6x + 5y + 17 = 0. \end{cases}$

(d) $\begin{cases} 7x + 11y = 35, \\ 8x - 3y = 21. \end{cases}$

(e) $\begin{cases} mx + ny + p = 0, \\ rx + sy + t = 0. \end{cases}$

(f) $\begin{cases} \dfrac{4}{5 + y} = \dfrac{5}{12 + x}, \\ 2x + 5y = 35. \end{cases}$

(g) $\begin{cases} \dfrac{3x + 4y + 3}{10} - \dfrac{2x + 7 - y}{15} = 5 + \dfrac{y - 8}{5}, \\ \dfrac{9y + 5x - 8}{12} - \dfrac{x + y}{4} = \dfrac{7x + 6}{11}. \end{cases}$

(h) $\begin{cases} \dfrac{5}{x} - \dfrac{7}{y} = 14, \\ \dfrac{7}{x} + \dfrac{8}{y} = 42. \end{cases}$

(i) $\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} = a, \\ \dfrac{p}{x} + \dfrac{q}{y} = b. \end{cases}$

[HINT. Solve for $\dfrac{1}{x}$ and $\dfrac{1}{y}$.]

2. Find two multipliers k_1 and k_2 such that when the equations of the system

$$\begin{cases} 3x + 4y = 20, \\ 5x + 8y = 17, \end{cases}$$

are multiplied by k_1 and k_2 respectively, and added, the coefficients of x and y in the new equation will be 4 and 0.

3. Two children weighing 25 and 40 pounds, respectively, are riding on a "teeter board" 14 feet long. How far should the support of the board be from the lighter child in order that they should just balance?

[HINT. The mechanical *principle of the lever* that is balanced is given by $w_1d_1 = w_2d_2$, where w_1 and w_2 are the weights and d_1d_2, their respective distances from the fulcrum.]

4. A grocer has two brands of coffee worth 25 and 40 cents a pound. How many pounds of each must he take to make a mixture of 100 pounds worth 35 cents a pound?

5. From Ex. 4 find the relative amounts of each brand regardless of the amount of the mixture.

6. Coffee worth a cents a pound is mixed with coffee worth b cents a pound to make a mixture worth c cents a pound. Give the rule for making the mixture.

7. A goldsmith wishes to mix 10 carat gold with 18 carat gold to make 20 ounces of 14 carat gold. How many ounces of each must be taken? [Pure gold is 24 carat.]

71. Graphic Solutions. In §§ 48 and 49 it was shown that the graph of a linear equation in two variables is a straight line. The system (1),

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

is therefore represented geometrically by *two straight lines*, one for each equation. In general, the two straight lines will intersect in a point and the coördinates of this point will satisfy both equations. The coördinates of the intersection, therefore, constitute a solution of the system. Consequently, for equations in two unknowns, the definition of a solution of a system given in § 69 may be translated into geometric language by substituting for the words *pair of values, one for each unknown*, the single word *point*, and for the words *satisfies all the equations of the system* the words *lies on all the graphs of the system.* For greater clearness the two definitions are given in parallel columns.

An algebraic solution of a system with two unknowns is a pair of values, one for each unknown, which satisfies all the equations of the system.

The graphic solution of a system is a point which lies on all the graphs of the system.

The approximate values of x and y are readily found by constructing the two graphs on coordinate paper and then reading from the paper the coordinates of the intersection.

For example, if it be required to solve graphically the system,

$$\begin{cases} 2x + 3y = 10, \\ 3x - 7y = 4, \end{cases}$$

the graphs may be constructed as in Fig. 19.

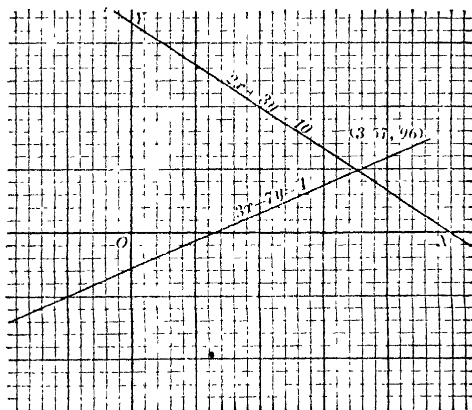


Fig. 19.

From the figure the coördinates of the point of intersection are seen to be approximately, $x = 3.6$, $y = 1$.

With coördinate paper ruled to tenths the solutions may be found accurately enough for most purposes.

EXERCISES

1. Solve graphically each of the following systems of equations, giving the results to the nearest tenth.

(a) $\begin{cases} 3x - 5y = 3, \\ 5x + 2y = 15. \end{cases}$

(b) $\begin{cases} 2x + 5y = 19, \\ 5x - 4y = 13. \end{cases}$

(c) $\begin{cases} 7x + 3y = 18, \\ x + 6y = 10. \end{cases}$

(d) $\begin{cases} 2x - 6y = 17, \\ x + 3y = -12. \end{cases}$

(e) $\begin{cases} x - y = 0, \\ 3x + 4y = 12. \end{cases}$

(f) $\begin{cases} x - 3 = 0, \\ 3x - 4y = 12. \end{cases}$

72. Consistent, Inconsistent, and Dependent Systems. Any attempt to solve the system of equations

$$\begin{cases} 2x + 3y = 5, \\ 6x + 9y = 7, \end{cases}$$

by the ordinary methods of elimination will lead to a contradiction. For example, it is possible to eliminate x by multiplying the first equation by 3 and subtracting the second. But y disappears along with x and nothing remains but the untrue statement, $15 - 7 = 0$. Systems of equations which, like the system just given, lead to a contradiction, are said to be ***inconsistent***. If the determinant solution of the system be examined, it will be seen that the determinant of the system,

$$\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix},$$

is zero. Since division by zero has no meaning (§ 14), it is reasonable to expect difficulty in such cases. The real nature of the difficulty is most easily discovered by going back to the general system (1)

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

whose solution is given in (2) and (3) in the form

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

If, now, $a_1b_2 - a_2b_1 = 0$, division by $-b_1b_2$ shows that

$$(6) \qquad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}.$$

But if b_1 and b_2 are different from zero, equations (1) may be written in the form

$$(7) \qquad \begin{cases} y = -\frac{a_1}{b_1}x + \frac{c_1}{b_1}, \\ y = -\frac{a_2}{b_2}x + \frac{c_2}{b_2}. \end{cases}$$

Since $-a_1/b_1$ and $-a_2/b_2$ are the slopes of the lines (§ 49), which represent equations (1), it follows from (6) and (7) that *if the determinant* $a_1b_2 - a_2b_1$ *is zero, the graphs of the system* (1) *are parallel,* and consequently do not meet in any finite point.

A pair of inconsistent linear equations in two unknowns is represented geometrically by two parallel lines.

If the numerators, as well as the denominators, of (2) and (3) are zero, the equations are said to be ***dependent.*** For dependent equations not only are the graphs parallel, but

$$(8) \qquad c_1b_2 - c_2b_1 = 0, \text{ and } a_1c_2 - a_2c_1 = 0.$$

The equations (8) are easily reduced to the forms

$$(9) \qquad \frac{c_1}{b_1} = \frac{c_2}{b_2}, \text{ and } \frac{a_1}{c_1} = \frac{a_2}{c_2},$$

which show that the intercepts of the two lines are equal. But if the slopes and the intercepts of two lines are respectively equal, the lines are coincident.

A pair of dependent equations in two unknowns is represented geometrically by a pair of coincident straight lines, that is, by a single straight line.

A system of equations which is neither inconsistent nor dependent is said to be ***consistent.*** Consistent systems have always a common solution.

EXERCISES

1. Determine in each of the following whether the equations are consistent, inconsistent, or dependent, and construct the graphs for each system.

(*a*) $\begin{cases} 2x + 7y = 5, \\ 4x + 14y = -5. \end{cases}$

(*b*) $\begin{cases} 3x + 15y = 12, \\ x + 5y = 4. \end{cases}$

(*c*) $\begin{cases} 3x - 6y = 4, \\ 4x - 8y = 4. \end{cases}$

(*d*) $\begin{cases} x - 2y = 3, \\ 2x + y = 5, \\ x + 8y = 1. \end{cases}$

2. Prove that when the equations (1) are dependent it is possible to find a constant k, which, used as a multiplier on one of the equations, will produce the other. Find k.

3. Can you recognize a system of two inconsistent or dependent equations in two unknowns without applying the determinant test? If so, how?

4. Prove that if in the system (1) we have $a_1b_2 - a_2b_1 = 0$ and $c_1b_2 - c_2b_1 = 0$, then also $a_1c_2 - a_2c_1 = 0$.

5. Prove that the lines determined by the three equations

$$Ax + By + C = 0, \qquad \frac{A}{D}x + \frac{B}{D}y + \frac{C}{D} = 0, \qquad MAx + MBy + MC = 0,$$

are identical, and from the conclusion deduce the proposition that multiplication or division of a linear equation by a constant does not affect the solutions of the equation, or the solution of any system of which it is a part.

6. Show by drawing a figure that a system of three equations in two unknowns is, in general, inconsistent.

7. Given a system of three linear equations in two unknowns,

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_3x + b_3y + c_3 = 0,$$

find the condition that the system shall be consistent by solving two of the equations for x and y and substituting the results in the third.

73. Linear Equations with Three Unknowns. The general form of a system of three linear equations in three unknowns is

$$(10)\qquad \begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

If the first equation be multiplied by $b_2c_3 - b_3c_2$, the second by $b_3c_1 - b_1c_3$, and the third by $b_1c_2 - b_2c_1$, and the resulting equations be added, the coefficients of y and z in the new equation will be zero. With both y and z eliminated we can find as the value of x

$$(11)\qquad x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}.$$

The denominator of this fraction is called a ***determinant of third order.*** It contains nine elements which may be arranged in the square array (12).

(12)

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} = D$$

By inspection of the terms in the denominator of x, it will be seen that the three positive terms are the products of elements lying on the arrows that start downward and to the right, while the three negative terms lie on the arrows ranging downward and to the left. D is called the ***determinant of the system.***

Moreover, it is clear that the numerator of x differs from the denominator only in the substitution of d_1, d_2, d_3 for a_1, a_2, a_3,

that is in the substitution of the absolute terms for the coefficients of x. Hence x can be found by Cramer's rule (§ 70).

It can be shown in a similar way that y and z may be found by the same rule so that the value of the three unknowns may be written down as follows:

$$(13)\quad x=\frac{\begin{vmatrix} d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},\quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},\quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}.$$

EXERCISES

1. Find the values of the unknowns in the following systems.

$$(a)\ \begin{cases} 3x+4y-5z=20,\\ x-3y+7z=30,\\ 5x+4y+z=40.\end{cases}\qquad (c)\ \begin{cases} x+2y-3z=8,\\ x-y=4,\\ y+z=6.\end{cases}$$

$$(b)\ \begin{cases} 4x+3y-2z-10=0,\\ 3x-2y+4z-20=0,\\ 5x+y-z+30=0.\end{cases}\qquad (d)\ \begin{cases} \dfrac{2}{x}+\dfrac{3}{y}-\dfrac{5}{z}=7,\\ \dfrac{1}{x}-\dfrac{2}{y}+\dfrac{4}{z}=6,\\ \dfrac{3}{x}-\dfrac{1}{y}+\dfrac{2}{z}=5.\end{cases}$$

2. Show that the multipliers $c_2a_3-c_3a_2$, $c_3a_1-c_1a_3$, and $c_1a_2-c_2a_1$ will enable us to find y as x was found at the beginning of this section.

3. Prove that the values of x, y, and z given in (13) are correct.

4. If x, y, z satisfy the system $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$, prove that

$$(14)\quad \frac{x}{\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}}=\frac{y}{\begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}}.$$

[HINT. Divide through by z, then solve for the ratios x/z and y/z.]

74. Applications. (1) *Determination of equations of given form belonging to loci which pass through given points.* For example, the equation of a straight line may be written in the form

$$y = ax + b,$$

but we know nothing about the line until the numbers a and b are known. From geometry we know that a line is determined by two points. If, therefore, any two points on the line are given, the numbers a and b may be found.

Suppose the line passes through the points (1, 2) and (2, 3). The fact that it passes through the point (1, 2) is expressed by the equation

$$2 = a \cdot 1 + b.$$

Similarly, the fact that it passes through (2, 3) is expressed by the equation

$$3 = a \cdot 2 + b.$$

From these two equations we readily find $a = 1$, $b = 1$. Consequently, the equation of the line passing through the two points (1, 2) and (2, 3) is

$$y = x + 1, \text{ or } x - y + 1 = 0.$$

(2) *Functions with unknown coefficients.* The law of motion of a body projected upward in a vacuum and subject to gravity is given by a function of the time having the form

$$s = -\tfrac{1}{2} gt^2 + vt,$$

where s is the altitude, g the acceleration due to gravitation, and v the velocity of projection.

An observer with a stop watch and a measuring line, notes that at the expiration of 1 second the body is at an altitude of 83.9 feet, and at the expiration of 6 seconds, at an altitude of 20.4 feet.

To find g and v we have the two equations

$$\begin{cases} 83.9 = -\frac{1}{2}g + v, \\ 20.4 = -18g + 6v. \end{cases}$$

From these equations

$$g = 32.2, \text{ and } v = 100 \text{ ft. per second.}$$

(3) *Alloys and chemical compounds.* From two lots of brass, one containing 2 parts copper to 1 part zinc, and the other containing 3 parts copper to 2 parts zinc, it is desired to make 60 pounds of brass that shall contain 5 parts copper to 3 of zinc. How much must be taken from each lot?

If x and y denote the number of pounds required of each kind

$$x + y = 60.$$

Moreover, 2/3 of the first lot, 3/5 of the second, and 5/8 of the mixture, are copper. Hence,

$$\frac{2}{3}x + \frac{3}{5}y = \frac{5}{8} \times 60 = \frac{75}{2}.$$

From these equations,

$$x = 22\tfrac{1}{2},\ y = 37\tfrac{1}{2}.$$

EXERCISES

1. The equation of a certain locus has the form

$$Ax^2 + By^2 = 11$$

and it passes through the two points (3, 4) and (2, 1). What are the values of A and B?

2. The equation of a circle in its most general form is

$$x^2 + y^2 + ax + by + c = 0.$$

How many points are necessary to determine a circle? How do you determine the number from the equation? What is the equation of the circle which passes through the points (2, 3) (5, − 2) (− 3, 4)?

3. Can a circle be drawn through the points (1, 2), (2, 4), and (5, 10)? Give the reason for your answer in algebraic language.

4. If we use the three numbers x, y, and z, to fix the position of a point in space with respect to three given planes, the symbol for a point is (x, y, z) and the equation

$$ax + by + cz + d = 0,$$

is the equation of a plane. Through how many points can a plane be made to pass? What is the equation of the plane passing through the points $(2, 3, 4)$, $(4, 3, 2)$, and $(0, 3, 4)$?

5. The force f of the pull required at one end of the rope of a pulley-block to raise a weight w, is given by the equation

$$f = a + b \cdot w.$$

If $f = 8.5$ lb. when $w = 100$ lb. and $f = 12.8$ lb. when $w = 200$ lb., find a and b.

6. The law for a body falling freely under gravity is expressed by

$$s = \tfrac{1}{2} gt + v_0 t + s_0$$

where s is space, $g = 32.2$, v_0 is velocity with which the body is started, and t is time in seconds. Two observations made at the expiration of 2 and 4 seconds, give $s = 204.4$ feet, and $s = 57.6$ feet, respectively. Find the unknown constants v_0 and s_0. Was the body thrown upward or downward? How do you know?

7. If the boiling point of water is $T°$ Fahrenheit at a height H above sea level,

$$H = A + BT + CT^2.$$

Find A, B, and C, if $T = 212$ when $H = 0$, $T = 211$ when $H = 516$, and $T = 210$ when $H = 1030$.

8. The electric resistance of platinum wire changes with the temperature, according to the equation

$$\frac{R_0}{R} = 1 - at + bt^2,$$

where R_0 is the resistance at zero degrees centigrade and R the resistance at temperature t. If $R_0 = 10$ and $R = 10.786664$ at $20°$ C. and $R = 11.569656$ at $40°$ C., what are the values of a and b?

9. The melting point $t°$ C. of an alloy of lead and zinc is given by a formula of the type

$$t = a + bx + cx^2,$$

where x is the percentage of lead in the alloy. Find a, b, and c if $t = 133$ when $x = 0$, $t = 233$ when $x = 50$, and $t = 333$ when $x = 100$.

10. Find the values of λ and μ such that $x^3 + 4x^2 + 2\lambda x + \mu$ shall be divisible by $x^2 + 2x + \lambda$.

11. If the expression

$$x^4 + 4x + 7\lambda x + 5\mu x + \rho$$

is divisible by

$$x^3 + 2x^2 + 3\lambda x + 2\mu,$$

find the values of λ, μ, and ρ.

CHAPTER VIII

SYSTEMS CONTAINING NON-LINEAR EQUATIONS

75. The Linear-Quadratic System. The simplest system of equations which are not all linear, consists of one linear and one quadratic equation. Such a system is called a ***linear-quadratic system*** and in its most general form is given by the two equations

$$(1)\qquad \begin{cases} ax + by + c = 0, \\ Ax^2 + By^2 + Cxy + Dx + Ey + F = 0. \end{cases}$$

The linear-quadratic system can always be solved by solving the linear equation for one of the unknowns and substituting the value thus obtained in the quadratic equation. The resulting equation will be a quadratic equation in one unknown which may be solved by the method of Chapter VI. When the value of one unknown has been found in this way the value of the other is easily found from the linear equation.

The linear-quadratic system has always *two* solutions, which, however, may not be distinct.

EXERCISES

1. Solve each of the following systems of equations.

(*a*) $\begin{cases} 3x + 4y = 7, \\ 2x^2 - 5y^2 = 0. \end{cases}$

(*b*) $\begin{cases} 2x - 5y = 6, \\ 3x^2 + 4y^2 = 7. \end{cases}$

(*c*) $\begin{cases} 3x - 6y + 4 = 0, \\ xy = 4. \end{cases}$

(*d*) $\begin{cases} 2x - y = 7, \\ 3x^2 + 4xy + 7y^2 = 12. \end{cases}$

(*e*) $\begin{cases} 8x + 4y - 9 = 0, \\ 3x^2 - 7xy + 4y^2 + 6x - 3y + 1 = 0. \end{cases}$

(*f*) $\begin{cases} 3x + 2y - 14 = 0, \\ 6x^2 - 4xy + 7y^2 - x + 3y = 0. \end{cases}$

2. In the system of equations

$$l = a + (n-1)d, \qquad s = \frac{n}{2}(a + l),$$

if a, d, and s are supposed to be known, find the unknowns.

3. In the system of example 2, if l, d, and s are supposed to be known, find the unknowns.

4. The sum of two numbers is 7 and the sum of the squares is 25. What are the numbers?

5. The sum of two numbers is 10 and the sum of their squares is 25. What are the numbers?

76. Geometric Interpretation of the Quadratic Equation in Two Variables.

A quadratic equation is represented geometrically by a curve called a ***conic section***, or more briefly, a ***conic***. A conic section is the curve formed by the intersection of a plane with a right cone. It belongs to one of three types. It is either an ***ellipse*** (of which the circle is a special case), a ***parabola***, or an ***hyperbola***.

(*a*) ***The circle.*** Let C be a fixed point in a plane with coördinates (a, b) (Fig. 20), and P a moving point with coördinates (x, y). Draw the ordinates MC and NP, and draw CR parallel to the x-axis meeting NP in R. The distance CP is found from the equation

$$\overline{CP}^2 = \overline{CR}^2 + \overline{RP}^2. \tag{2}$$

From the figure $CR = x - a$, and $RP = y - b$. If the distance CP be denoted by r, we have

$$r^2 = (x-a)^2 + (y-b)^2,$$

which is the formula for the distance between the two points (a, b) and (x, y). If, therefore, the point P moves in such a way that its distance from C is constant it will trace out a

circle. The coördinates of every point on this circle will satisfy the equation

(3) $$(x-a)^2+(y-b)^2=r^2.$$

Conversely, every equation which can be reduced to the form (3) is the equation of a circle, for (3) states that the distance between a moving point (x, y) and a fixed point (a, b) is always constant.

Any equation which has the form

(4) $$x^2+y^2+Ax+By+C=0$$

can be reduced to the form (3).

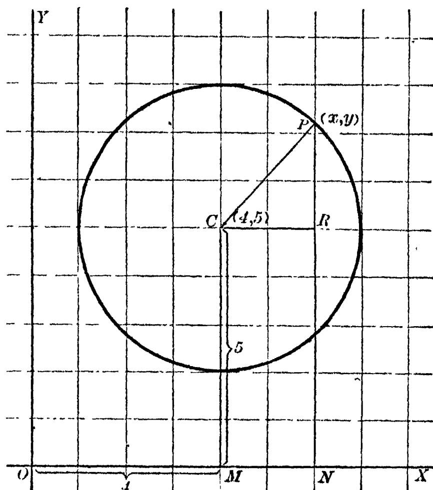


FIG. 20.

For example, the equation

$$x^2+y^2-8x-10y+32=0$$

may be reduced first to the form

$$x^2-8x+y^2-10y=-32,$$

and then, by completing the square for the terms in x and for the terms in y, to the form

$$(x-4)^2+(y-5)^2=9.$$

The equation is therefore the equation of a circle whose center is at the point (4, 5) and whose radius is 3.

When a and b are both zero, equation (3) takes the form

(5) $$x^2+y^2=r^2,$$

which is the equation of a circle of radius r and with its center at the origin.

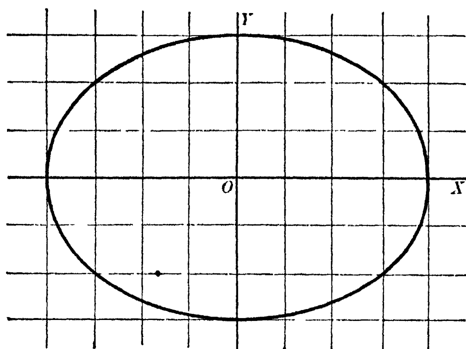


FIG. 21.

(*b*) ***The ellipse.*** In books on analytic geometry it is shown that by proper choice of axes and origin, the equation of any ellipse can be written in the form

(6) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b are constants. The curve is an oval as shown in Fig. 21.

(*c*) ***The parabola.*** As in the case of the ellipse it may be shown that the equation of a parabola can be written in the form

(7) $$y^2 = 2\,px,$$

where p is a constant. The parabola

$$y^2 = 4\,x$$

is shown in Fig. 22.

Similarly, the equation

$$x^2 = 2\,py$$

represents a parabola which is symmetric about the y-axis. (See p. 65.)

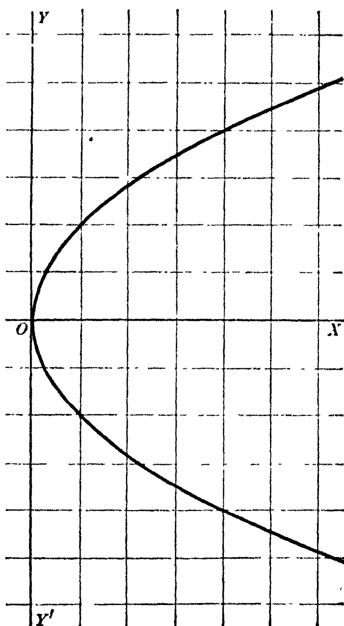


FIG. 22.

(*d*) ***The Hyperbola.*** The equation of an hyperbola may be written in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{8}$$

where a and b are constants. The curve has two parts as shown in Fig. 23.

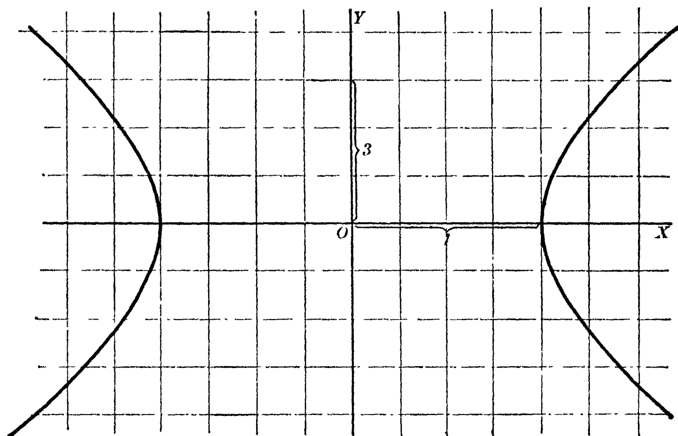


FIG. 23.

If a and b in equation (8) are equal, the equation of the hyperbola may be written in the form

$$xy = c, \tag{9}$$

where c is a constant. When the equation is written in the form (9), the curve has the position as shown in Fig. 13, § 54.

The conic sections play a very important part in both pure and applied mathematics. The graph of every equation of the second degree in two variables is a conic section. The curves themselves have an important place in mechanics and astronomy. For example, the path of a body moving freely under the influence of an attracting body is a conic section. Consequently, the paths of the planets about the sun are conic sections.

The ***intercepts*** of a curve are the distances from the origin to the points in which the curve cuts the axes. The x-intercepts are found by putting y equal to zero and solving for x, and the y-intercepts are found by making x equal to zero and solving for y.

When the equation of the conic is not in one of the standard forms (4), (5), (6), (7), (8), or (9), the curve may be constructed by solving the equation for y and then making a table of values as in § 46. It is important to note that for each value of x there will be, in general, *two* values of y on account of the radical sign that enters.

EXERCISES

1. Find the distances between each of the following pairs of points.

(*a*) $(0, 0)$ and $(3, 4)$.
(*b*) $(1, 1)$ and $(2, 4)$.
(*c*) $(1, 0)$ and $(5, 6)$.
(*d*) $(-1, 7)$ and $(1, -1)$.
(*e*) (x, y) and $(c, 0)$.
(*f*) (x, y) and $(-c, 0)$.

2. Find the equations of the circles with centers and radii as follows.

(*a*) Center $(5, 6)$, radius 7.
(*b*) Center $(5, -7)$, radius 4.
(*c*) Center $(5, -7)$, radius 4.
(*d*) Center $(-2, -3)$, radius 6.
(*e*) Center $(-5, 4)$, radius 3.
(*f*) Center $(1, 2)$, radius 0.
(*g*) Center $(0, 0)$, radius 0.

3. Locate the center and find the radius of each of the following circles.

(*a*) $x^2 + y^2 - 6x + 8y - 24 = 0$.
(*b*) $x^2 + y^2 + 6x + 8y - 24 = 0$.
(*c*) $x^2 + y^2 - 6x + 4y - 30 = 0$.
(*d*) $x^2 + y^2 - ax + by - c = 0$.
(*e*) $x^2 + y^2 + 4x + 6y + 20 = 0$.

4. Construct the graphs of the following equations, giving the intercepts and interpreting any imaginary results that occur.

(*a*) $x^2 + y^2 = 36$.
(*b*) $5x^2 + 9y^2 = 45$.
(*c*) $5x^2 - 9y^2 = 45$.
(*d*) $x^2 + y^2 - 2x + 3y + 1 = 0$.
(*e*) $x^2 + y^2 - 2x - 4y - 10 = 0$.
(*f*) $xy = 6$.

77. The Graphical Solution of the Linear-Quadratic System. The graph of the linear equation is a straight line and that of the quadratic is a conic. It is a fundamental property of a conic that it may be cut by a straight line in at most two real points. If there are two intersections, the system has two real solutions; if the lines do not intersect, there are two imaginary solutions.

EXERCISES

1. Solve each of the following systems, and interpret the results by constructing the graphs.

(*a*) $\begin{cases} x + y = 4, \\ x^2 + y^2 = 16. \end{cases}$

(*b*) $\begin{cases} x + y = 4, \\ x^2 - y^2 = 16. \end{cases}$

(*c*) $\begin{cases} 3x + 4y + 7 = 0, \\ y^2 - 7x = 0. \end{cases}$

(*d*) $\begin{cases} 2x + 3y = 6, \\ 16x^2 + 25y^2 = 400. \end{cases}$

2. One point moves in such a way that its distance from the y-axis is 3 times its distance from the x-axis, and another moves along the circle whose radius is 5 and whose center is at the origin. Where do the two paths cross?

3. Find the value of k such that the straight line $y = 2x + k$ will just touch the circle $x^2 + y^2 = 25$. What does the double sign for the radical in your answer mean?

78. Systems Linear in x^2 and y^2. Such a system is necessarily of the form

$$(10) \qquad \begin{cases} a_1x^2 + b_1y^2 = c_1, \\ a_2x^2 + b_2y^2 = c_2. \end{cases}$$

From this system the values of x^2 and y^2 may be found as in

systems of linear equations. The values of x and y are then found by a simple root extraction.

In this case there are always *four* solutions, which are obtained by using all possible combinations of signs for x and y.

The system is represented graphically by two conics which

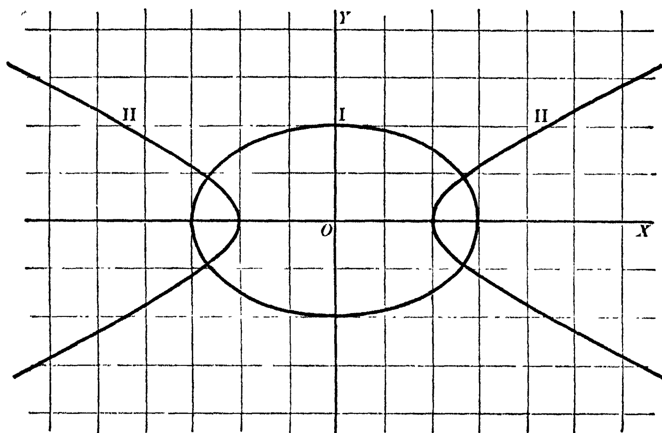


Fig. 24.

may intersect in four points, or in two points, or not at all (Fig. 24). What is the character of the solutions in each of the three cases?

EXERCISES

1. Solve each of the following systems of equations.

(a) $\begin{cases} 5x^2 + 7y^2 = 8, \\ 4x^2 - 6y^2 = 9. \end{cases}$ (c) $\begin{cases} 8y^2 - 4x^2 - 11 = 0, \\ 8x^2 + 4y^2 - 11 = 0. \end{cases}$ (e) $\begin{cases} 3x^2 + 14y^2 = 63, \\ x^2 + y^2 = 9. \end{cases}$

(b) $\begin{cases} 6x^2 + 14y^2 - 7 = 0, \\ 3x^2 + 8y^2 - 9 = 0. \end{cases}$ (d) $\begin{cases} 8x^2 - 6y^2 = 15, \\ 5x^2 + 3y^2 = 15. \end{cases}$ (f) $\begin{cases} 2x^3 + 3y^3 = 100, \\ 3x^3 - 4y^3 = 75. \end{cases}$

2. If the two conics

$$4x^2 + 9y^2 = k \text{ and } x^2 - 4y^2 = 4$$

just touch each other, what is the value of k, and in what points do the conics touch?

79. Systems of Symmetric Equations. An equation in two unknowns is *symmetric* if it remains unaltered when the unknowns are interchanged. The most general form of a system of symmetric quadratic equations is

$$\text{(11)} \qquad \begin{cases} x^2+y^2+a_1xy+b_1(x+y)=c_1, \\ x^2+y^2+a_2xy+b_2(x+y)=c_2. \end{cases}$$

A symmetric system may be solved by first making the substitution

$$\text{(12)} \qquad x=u+v, \quad y=u-v.$$

The original equations then become

$$\text{(13)} \qquad \begin{cases} (2+a_1)u^2+(2-a_1)v^2+2\,b_1u=c_1, \\ (2+a_2)u^2+(2-a_2)v^2+2\,b_2u=c_2. \end{cases}$$

When v^2 is eliminated from equations (13), a single quadratic equation in u is obtained. This equation can be solved for u, and either root substituted in one of the equations (13) will give another quadratic in v from which v may be found. When u and v are both known, x and y may be found from equations (12). There are *four* solutions.

EXERCISES

1. Solve each of the following systems of equations.

(a) $\begin{cases} x^2+y^2+3\,xy+4(x+y)=23, \\ x^2+y^2-4\,xy+2(x+y)=3. \end{cases}$

(b) $\begin{cases} 2(x^2+y^2)+3\,xy-5(x+y)=19, \\ 3(x^2+y^2)-2\,xy+4(x+y)=47. \end{cases}$

(c) $\begin{cases} 5\,xy+6(x+y)=154, \\ 2(x^2+y^2)+5(x+y)=127. \end{cases}$

(d) $\begin{cases} 3(x^2+y^2)-6\,xy+12(x+y)=135, \\ 4(x^2+y^2)+7\,xy+13(x+y)=597. \end{cases}$

(e) $\begin{cases} xy=45, \\ 3(x^2+y^2)-6\,xy+11(x+y)=17. \end{cases}$

(f) $\begin{cases} x^2+y^2=25, \\ 2\,x^2+6\,xy+2\,y^2-7(x+y)=22. \end{cases}$

2. Find multipliers k_1 and k_2 for equations (11) such that when the multiplied equations are added, the result will be a single equation of the form

$$x^2 + y^2 + 2xy + (k_1b_1 + k_2b_2)(x + y) = k_1c_1 + k_2c_2.$$

3. Solve the system

$$\begin{cases} x^2 + y^2 + 4xy + 6(x + y) = 31, \\ x^2 + y^2 + 8xy - 4(x + y) = 9, \end{cases}$$

by finding an equation similar to that found in Ex. 2, and then finding from this equation two linear equations.

80. Systems with all Terms of Degree 2 or 0. Such a system has the form

$$\begin{cases} a_1x^2 + b_1y^2 + c_1xy = d_1, \\ a_2x^2 + b_2y^2 + c_2xy = d_2. \end{cases} \tag{14}$$

To solve this system, let $y = vx$, and find

$$\begin{cases} (a_1 + b_1v^2 + c_1v)x^2 = d_1, \\ (a_2 + b_1v^2 + c_2v)x^2 = d_2. \end{cases} \tag{15}$$

The elimination of x^2 leads to an equation in v from which two values, v_1 and v_2, are found for v.

Either one of the equations $y = v_1x$ or $y = v_2x$ together with one of the original equations forms a linear quadratic system from which x and y are readily found. There are *four* solutions. Why?

EXERCISES

1. Solve each of the following systems of equations.

(*a*) $\begin{cases} 3x^2 + 5y^2 + 4xy = 31, \\ 5x^2 + 4y^2 - 6xy = 9. \end{cases}$

(*b*) $\begin{cases} 2x^2 - 6y^2 + 14xy = 30, \\ 3x^2 + 7y^2 - 14xy = -9. \end{cases}$

(*c*) $\begin{cases} 3x^2 - 6y^2 + 5xy = -12, \\ 4x^2 + 7y^2 - xy = 73. \end{cases}$

(*d*) $\begin{cases} 3x^2 - 7y^2 + 8xy = 73, \\ 4x^2 + xy = 20. \end{cases}$

(*e*) $\begin{cases} 6y^2 - 5xy = 66, \\ 3x^2 + 4xy = 228. \end{cases}$

(*f*) $\begin{cases} 2x^2 - 7xy = 18, \\ 3x^2 + 5y^2 - xy = 42. \end{cases}$

2. In either of the equations (14) $-x$ and $-y$ may be substituted for $+x$ and $+y$ respectively without altering the equation. What is the geometric significance of this fact?

81. Systems solved by Various Devices. No solution of the most general system of simultaneous quadratic equations in two unknowns can be given at the present time since such a solution involves the solution of an equation of the fourth degree. It is possible, however, to find solutions for many special systems of quadratics as well as systems involving equations of higher degrees.

EXERCISES

1. Solve each of the following system of equations.

$(a)\ \begin{cases} x^2 + y^2 + x + y = 330, \\ x^2 - y^2 + x - y = 150. \end{cases}$ $\qquad (e)\ \begin{cases} \sqrt{x + y} + x + y = 12, \\ x^2 + y^2 = 41. \end{cases}$

$(b)\ \begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$ $\qquad (f)\ \begin{cases} x^3 - y^3 = 98, \\ x - y = 2. \end{cases}$

$(c)\ \begin{cases} x^2y^2 = 180 - xy, \\ x + 3\,y = 11. \end{cases}$ $\qquad (g)\ \begin{cases} x^3 + y^3 = 407, \\ x + y = 11. \end{cases}$

$(d)\ \begin{cases} xy + xy^2 = 12, \\ x + xy^3 = 18. \end{cases}$ $\qquad (h)\ \begin{cases} x^3 + y^3 = 152, \\ x^2 - xy + y^2 = 19. \end{cases}$

2. After appropriate substitution of new letters, solve each of the following systems of equations.

$(a)\ \begin{cases} x^{1/3} + y^{1/3} = 5, \\ x + y = 35. \end{cases}$ $\qquad (c)\ \begin{cases} x + y = 4, \\ x^4 + y^4 = 82. \end{cases}$

$(b)\ \begin{cases} x^{1/2} + y^{1/2} = 4, \\ x^{3/2} + y^{3/2} = 28. \end{cases}$ $\qquad (d)\ \begin{cases} x^{1/4} + y^{1/3} = 5, \\ x^{1/2} + y^{2/3} = 13. \end{cases}$

3. Could you have foretold the symmetric form of the results in problems $2(a) - 2(c)$?

4. Prove that the elimination of y from the system

$$\begin{cases} x^2 + y^2 - 2\,x = 4\,y - 20 = 0, \\ \quad x^2 + 25\,y^2 = 25, \end{cases}$$

leads to an equation with rational coefficients of degree 4 in x. What is the geometric meaning of the result?

5. Prove by means of two equations each having the form

$$x^2 + y^2 + ax + by + c = 0,$$

that two circles cannot intersect in more than two real points.

[Hint. Subtract one equation from the other. The intersections will lie on the graph of the resulting equation.]

82. The Graphic Solution of Quadratic Equations. The accurate construction of the graph of a quadratic function in itself constitutes a graphic solution of the corresponding quadratic equation when the roots are real. Another solution more in harmony with the methods for the solution of equations of the third and fourth degrees is given here.

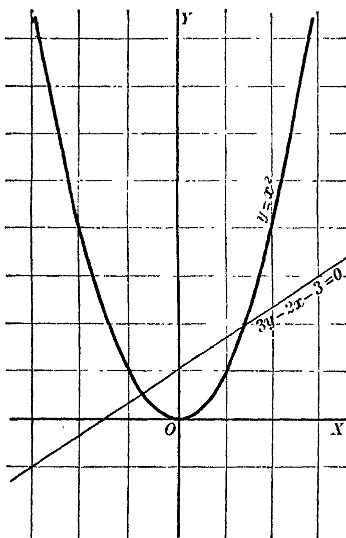


FIG. 25.

Let the quadratic equation be

(16) $\quad ax^2 + bx + c = 0,$

and let

$$y = x^2.$$

The equation (16) is then equivalent to the system

(17) $\begin{cases} y = x^2, \\ ay + bx + c = 0. \end{cases}$

When the roots are real, their approximate values may be read off directly from the intersections of the parabola $y = x^2$ and the straight line $ay + bx + c = 0$. If the roots are imaginary the fact is apparent at once from the figure.

EXAMPLE. Let it be required to solve graphically the quadratic equation

$$3x^2 - 2x - 3 = 0.$$

The solution is found from the intersections of the parabola $y = x^2$ and the straight line $3y - 2x - 3 = 0$. The approximate values as read off from Fig. 25 are $x_1 = 1.48$ and $x_2 = -.83$.

The parabola must, of course, be drawn with care. This can be done by plotting the points for fractional values of x. With a good copy of the parabola cut out of cardboard, and a ruler, graphic solutions of quadratic equations may be found rapidly.

EXERCISES

1. Solve graphically each of the following equations.

(a) $x^2 - 4x + 1 = 0.$

(b) $x^2 - 4x - 3 = 0.$

(c) $4x^2 - 8x - 3 = 0.$

(d) $9x^2 - 36 + 23 = 0.$

(e) $x^2 - 6x + 8.9 = 0.$

(f) $x^2 - 10x + 25 = 0.$

(g) $x^2 - 4x - 97 = 0.$

(h) $2x^2 - 8x - 431 = 0.$

83. Graphic Solution of the Incomplete Cubic. Any cubic equation is easily reduced, by a process that will be explained in § 100, to the so-called *incomplete form*,

$$rx^3 + px + q = 0. \tag{18}$$

If $y = x^3$, this equation is equivalent to the system

$$\begin{cases} y = x^3, \\ ry + px + q = 0. \end{cases} \tag{19}$$

The graph of the first equation is the graph of the power function x^3 and the graph of the second is a straight line. The approximate values of the roots are easily read off from the intersections of the two lines.

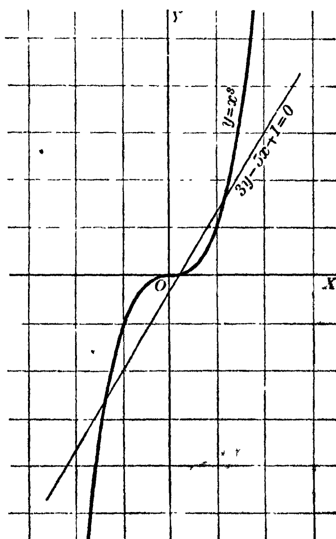


FIG. 26.

EXAMPLE. Solve graphically the equation

$$3x^3 - 5x + 1 = 0.$$

This equation is equivalent to the system

$$\begin{cases} y = x^3. \\ 3y - 5x + 1 = 0. \end{cases}$$

From Fig. 26 the approximate values of x are read off to be

$$x_1 = -1.3,\quad x_2 = .2,\quad x_3 = 1.2.$$

EXERCISES

1. Solve graphically each of the following equations.

(*a*) $3x^3 - 5x + 2 = 0.$ (*c*) $3x^3 - 5x + 5 = 0.$

(*b*) $3x^3 + 5x + 2 = 0.$ (*d*) $60x^3 + 115x - 56 = 0.$

84. Graphic Solution of the Incomplete Biquadratic. An equation of the fourth degree is called a ***biquadratic.*** In a later section (§ 100) it will be shown that every biquadratic can be reduced to the *incomplete form*,

(20) $$x^4 + px^2 + qx + r = 0;$$

or, by a slight rearrangement of the terms of equation (20), to the form

(21) $$x^2 + x^4 + (p-1)x^2 + qx + r = 0.$$

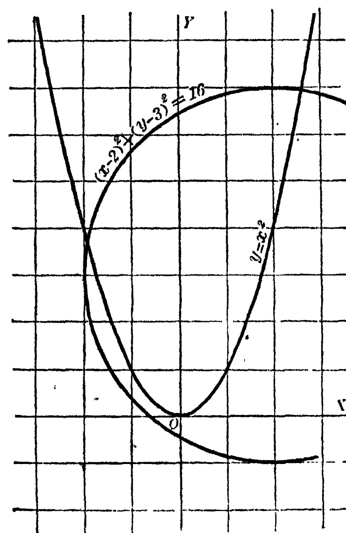


Fig. 27.

If y be chosen so that $y = x^2$, the equation becomes equivalent to the system

(22) $$\begin{cases} y = x^2, \\ x^2 + y^2 + qx + \\ (p-1)\,y + r = 0. \end{cases}$$

The graph of the first equation is a parabola and that of the second is always a circle. The real roots will be read off from the intersections of the graphs.

Example. Solve graphically the equation

$$x^4 - 5x^2 - 4x - 3 = 0.$$

The term $-5x^2$ may be written in the form $-6x^2 + x^2$.

Then if $y = x^2$, the equation becomes equivalent to the system.

$$y = x^2,\ x^2 + y^2 - 6\,y - 4\,x - 3 = 0.$$

When the squares of the terms in x and of the terms in y are completed the second equation takes the form

$$(x - 2)^2 + (y - 3)^2 = 16.$$

In this form the graph is readily seen to be a circle of radius 4, and with its center at the point (2, 3). From Fig. 26, the real values of x are seen to be $x_1 = -2$, and $x_2 = 2.6$. Two roots are imaginary.

EXERCISES

1. Solve graphically the following equations.

(*a*) $x^4 - 5\,x^2 - 4\,x - 7 = 0.$

(*b*) $x^4 - 5\,x^2 - 2\,x - 27 = 0.$

(*c*) $x^4 + 7\,x^2 - 2\,x - 38 = 0.$

(*d*) $2\,x^4 + 6\,x^2 - 8\,x - 41 = 0.$

CHAPTER IX

INEQUALITIES

85. Definitions and Theorems. An ***inequality*** is a statement which affirms that one number, or one quantity, is greater than another. If a is greater than b the fact is expressed by writing

$$a > b, \text{ or } b < a.$$

The opening of the inequality sign is always turned toward the greater number. Two inequalities such as

$$a > b \text{ and } x^2 > 5x - 6,$$

which have the inequality sign turned in the same direction, are said to exist in the *same sense*. It is everywhere assumed that the numbers in the inequalities are real. The rules of operation for inequalities are, with two important exceptions, the same as the rules of operation for identities, as the following theorems show.

THEOREM I. *An inequality with constant terms remains true in the same sense if any number, either positive or negative, is added to both sides.*

In symbols, the theorem states that if $a > b$, then

$$a \pm m > b \pm m.$$

If $a > b$, $a = b + d$, where d is the difference between a and b. Therefore

$$a + m = b + m + d.$$

If the positive number d be dropped from the right member, the equation becomes the inequality

$$a + m > b + m.$$

Corollary. *Any number may be transposed from one side of an inequality to the other if, at the same time, its sign is changed.*

Theorem II. *An inequality remains true in the same sense if both sides are multiplied or divided by the same positive constant.*

In symbols, the theorem states that if $a > b$, then $ma > mb$, where m is a positive number.

As in Theorem I, $a = b + d$ where d is a positive number; then $ma = mb + md$. If the positive number md be dropped, the equation $ma = mb + md$ becomes the inequality $ma > mb$.

That the theorem is true for division is clear if we note that the theorem is true for $m = 1/n$, where n is positive.

Theorem III. *The sense of an inequality is changed if both sides be multiplied or divided by a negative number.*

For, if $a > b$, then $-ma = -mb - md$, as before. Discarding the negative number $-md$ from the right member of the equation increases that member and the result is the inequality $-ma < -mb$.

If both sides of an inequality are positive, they may be raised to any positive integral power and the result will be a new inequality having the same sense. Every numerical inequality can be changed in form by transposition so that both sides will be positive numbers.

The *solution* of an inequality with one unknown is the process of finding all the values of the unknown for which the inequality is true.

EXERCISES

1. Prove that $a^2 + b^2 > 2ab$ for every real value of a and b, unless $a = b$.

[Hint. $(a - b)^2 > 0$, unless $a - b = 0$.]

2. Prove that when a and b have the same sign $a/b + b/a > 2$, unless $a = b$.

3. Prove that $a^2 + b^2 + c^2 > 2ab + 2bc + 2ac$ for real values of a, b, and c.

4. Prove the following inequalities.

(*a*) $\sqrt{10} > \sqrt{2} + \sqrt{3}$.

(*b*) $\sqrt{11} < \sqrt{3} + \sqrt{4}$.

(*c*) $\sqrt{14} > 2 + \sqrt{3}$.

(*d*) $\sqrt{2} + \sqrt{5} > \sqrt{3} + \sqrt{4}$.

(*e*) $\sqrt{12} - \sqrt{3} > \sqrt{11} - \sqrt{5}$.

(*f*) $\sqrt{11} - \sqrt{6} + \sqrt{5} < \sqrt{10}$.

5. Of the following pairs of numbers which is greater,

(*a*) $\sqrt{3}$ or $2 + \sqrt{3}$?

(*b*) $\sqrt{21} + \sqrt{10}$ or $\sqrt{17} + \sqrt{13}$?

(*c*) $\sqrt{8} - \sqrt{7}$ or $\sqrt{3} - \sqrt{2}$?

(*d*) $\sqrt{21} - \sqrt{17} + \sqrt{3}$ or 2.2?

6. For what values of x is the inequality $5x - 3 > 3x - 7$ true?

Algebraic Solution. By means of the theorems on inequalities this inequality is easily reduced to the form $x + 2 > 0$, and from this form we see that x must be greater than -2.

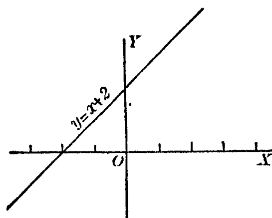


Fig. 28

Graphical Solution. A graphical solution is easily obtained by drawing the graph of the function $x + 2$. The graph is a straight line crossing the x-axis at $x = -2$ as shown in Fig. 28. The graph lies above the x-axis for all values of x that are greater than -2. In other words the function $x + 2$ is greater than zero for $x > -2$.

7. Solve the inequality

$$x^2 < 5x - 6.$$

Solution. The inequality is easily reduced to the form

$$x^2 - 5x + 6 < 0,$$

or $$(x-2)(x-3) < 0.$$

The graph of the left member of the last form is a parabola which crosses the x-axis at $x=2$ and $x=3$. The figure (Fig. 29) shows that the function is less than zero, that is, the original inequality is satisfied for values of x lying between 2 and 3. The solution may be written in the form

$$2 < x < 3.$$

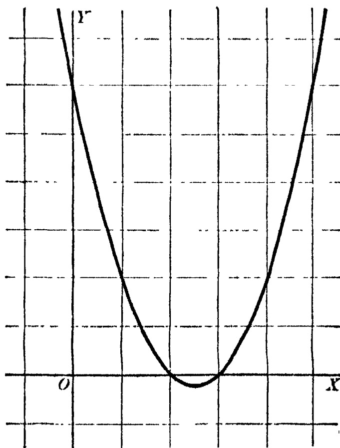


Fig. 29.

8. Find the values of x which satisfy the following inequalities and construct the graph in each case.

(*a*) $x^2 - 5x + 6 > 0$.

(*b*) $-x^2 > 6 - 5x$.

(*c*) $x^2 + x > 1$.

(*d*) $x^2 + x + 1 < 0$.

(*e*) $(x-1)(x-2)(x-3) > 0$.

(*f*) $(x-1)(x-2)(x-3) < 0$.

9. Prove that the inequality $ax^2 + bx + c > 0$ is true for all values of x if $b^2 - 4ac < 0$ and a is positive.

86. Inequalities with Two Unknowns. The solution of an inequality with two unknowns consists in finding all pairs of values of the unknowns for which the inequality is true. For example, to solve the inequality

$$2x + 3y - 5 > 0,$$

note that the line whose equation is

$$2x + 3y - 5 = 0,$$

separates all the points of the plane into three classes:

(1) Points whose coördinates satisfy the equation

$$2x + 3y - 5 = 0;$$

(2) points whose coordinates satisfy the inequality

$$2x + 3y - 5 > 0;$$

(3) points whose coördinates satisfy the inequality

$$2x + 3y - 5 < 0.$$

The points in the first class lie on the line. If the equation be written in the form

$$y = \frac{5 - 2x}{3},$$

it becomes clear that the points of the second class lie above the line, for if x is unchanged and y increased from y to y', we have

$$y' > \frac{5 - 2x}{3}.$$

Similarly, it may be shown that points of the third class lie below the line.

In the same way, if we consider the line whose equation is

$$3x - 4y + 5 = 0,$$

it may be shown that points whose coördinates satisfy the inequality $3x - 4y + 5 > 0$ lie *below* the line, while points whose coördinates reverse the sense of the inequality lie *above* the line.

Finally, the points which satisfy the two simultaneous inequalities

$$\begin{cases} 2x + 3y - 5 > 0, \\ 3x - 4y + 5 > 0, \end{cases}$$

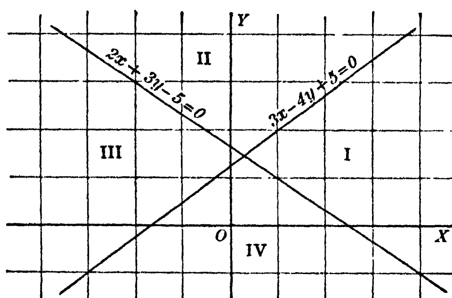


Fig. 30.

must lie above the first line and below the second. They must therefore lie in the region I of Fig. 30.

EXERCISES

1. Solve graphically each of the following systems of inequalities.

(*a*) $\begin{cases} 2x + 3y - 7 > 0, \\ -x + 2y + 3 > 0. \end{cases}$

(*b*) $\begin{cases} 2x + 3y - 7 > 0, \\ -x + 2y + 3 < 0. \end{cases}$

(*c*) $\begin{cases} 2x + 3y - 7 > 0, \\ -x + 2y + 3 > 0. \end{cases}$

(*d*) $x^2 + y^2 - 16 > 0.$

(*e*) $\begin{cases} 2x + 3y - 7 > 0, \\ x - y - 1 > 0, \\ x - 3 > 0. \end{cases}$

(*f*) $x^2 + y^2 - 16 < 0.$

(*g*) $\begin{cases} x^2 + y^2 - 16 < 0, \\ x + y - 2 > 0. \end{cases}$

(*h*) $\begin{cases} x^2 - y^2 - 16 > 0, \\ 2x + y - 1 > 0. \end{cases}$

CHAPTER X

COMPLEX NUMBERS

87. Definitions. The solution of the quadratic equation

$$x^2 + 1 = 0$$

is given by $x = \pm\sqrt{-1}$, or, as it is usually expressed, $x = \pm\, i$, where i is a symbol used to denote $\sqrt{-1}$. The number denoted by i cannot be real, for the square of every real number, either positive or negative, is positive, while

$$i^2 \equiv -1.$$

Numbers of the form bi, where b is a real number, are called ***pure imaginaries*** (§ 33).

Assuming that i obeys all the laws of ordinary algebra, including the Index Law, the powers of i are given by the following table:

$$i^1 = i,$$
$$i^2 = -1,$$
$$i^3 = i^2 i = -i,$$
$$i^4 = i^2 i^2 = +1,$$
$$i^5 = i^4 i = i,$$
$$\cdots\cdots\cdots\cdots$$

Moreover,

$$(bi)^2 = bi \cdot bi = b^2 i^2 = -b^2.$$

Numbers having the form $a + bi$, where a and b are real, are called ***complex numbers.***

Since bi is not real, it follows that if $a + bi = 0$, then $a = 0$, and $b = 0$, since, otherwise, $bi = -a$, that is, a pure imaginary is equal to a real number, which is impossible. From this it follows that two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are

equal, when, and only when, the coefficients of the real and of the imaginary parts are separately equal. For, if we have

$$a_1 + b_1 i = a_2 + b_2 i,$$

then

$$a_1 - a_2 + (b_1 - b_2) i = 0,$$

and consequently,

$$a_1 = a_2 \text{ and } b_1 = b_2.$$

88. Geometric Representation. The complex number $a + bi$, or $a \cdot 1 + b \cdot i$, contains two fundamentally different units, 1 and i. In the geometric representation of complex numbers this difference is recognized by measuring the real part along one axis called the ***axis of reals***, and the imaginary part along another axis perpendicular to it and called the ***axis of imaginaries***. It is agreed that the axis of reals shall coincide with the x-axis and the axis of imaginaries shall coincide with the y-axis. On the basis of this agreement, the complex number $a + bi$ is represented, either by a broken line measured a units along the axis of reals, then b units along the axis of imaginaries, or by the point P reached by the process indicated. Figure 31 shows the geometric representa-

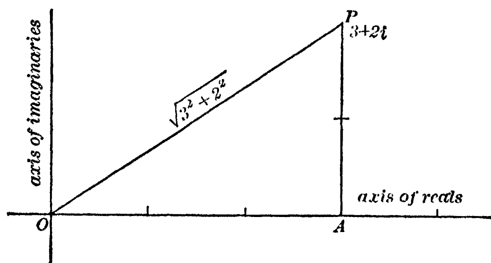


FIG. 31.

tion of $3 + 2i$. The numbers $3 + 2i$, $-3 + 2i$, $-3 - 2i$, and $3 - 2i$ are represented by numbers lying in the first, second, third, and fourth quadrants respectively.

The positive number $\sqrt{a^2+b^2}$ which measures the distance OP is called the ***absolute value*** or the ***modulus*** of the complex number $a+bi$. The absolute value of a complex number is denoted by $|a+bi|$ or by *mod* $(a+bi)$. The angle θ measured counter-clockwise from the axis of reals to the line OP, is called the ***angle*** or the ***amplitude*** of the complex number.

The complex number $a-bi$ is called the ***conjugate*** of the complex number $a+bi$. A complex number and its conjugate have the same absolute value. If a complex number is zero, its absolute value is zero, and conversely.

EXERCISE

Prove that the absolute value of a real number is identical with its numerical value.

89. The Fundamental Operations on Complex Numbers.

THEOREM. *The sum, the difference, the product, and the quotient of two complex numbers are complex numbers expressible in the standard form* $a+bi$.

Let a_1+b_1i and a_2+b_2i be two complex numbers.

Their sum is

$$a_1+b_1i+a_2+b_2i=a_1+a_2+(b_1+b_2)i.$$

Their difference is

$$a_1+b_1i-(a_2+b_2i)=a_1-a_2+(b_1-b_2)i.$$

Their product is

$$(a_1+b_1i)(a_2+b_2i)=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i.$$

Their quotient is

$$\frac{a_1+b_1i}{a_2+b_2i}=\frac{(a_1+b_1i)(a_2-b_2i)}{(a_2+b_2i)(a_2-b_2i)}$$

$$=\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^2+b_2^2}$$

$$=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}i.$$

The final result in each case is a complex number of the standard form, and the theorem is therefore proved. That the apparent exceptions where the result is either real or pure imaginary are not really exceptions is shown by the fact that a real number may be written in the form $a + 0 \cdot i$, while a pure imaginary takes the form $0 + bi$.

The geometric representation of a complex number is most easily found by first reducing the number to the standard form.

EXERCISES

1. Give the geometric representation and determine the absolute value of each of the following numbers.

(a) $2 + 3i$.

(b) $2 - 3i$.

(c) $4(2 + 3i)$.

(d) 2.

(e) -2.

(f) $-i$.

(g) $\frac{1}{i}$.

(h) $(3 - 4i)^2$.

(i) $\frac{2 + 3i}{3 - 4i}$.

(j) $(1 + i)^3$.

(k) $(1 - i)^3$.

(l) $(6 + \frac{1}{2}i)^2$.

2. Prove that the sum and the product of a complex number and its conjugate are real. What of their difference?

3. Construct the graph of $z^2 + 2z + 3$, when $z = 1 + 2i$.

4. Construct the graph of $z^2 + 2z + 3$, when $z = 1 - 2i$.

5. Find the real and imaginary parts of $f(a + bi)$ and $f(a - bi)$ if

$$f(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4,$$

and the a's are all real. What difference in the results do you note?

6. Prove that if $f(z)$ is an integral quadratic function of z with real coefficients, that $f(a + bi)$ has the form $A + Bi$ and $f(a - bi)$ has the form $A - Bi$. Is the result true for an integral function of any degree?

7. By means of the absolute value, prove that if $a + bi = 0$, then $a = 0$ and $b = 0$, and conversely.

90. Geometric or Vector Addition. In § 88 it was shown that the complex number $a + bi$ may be represented geometrically either by the broken line OAP, or by the point P

(Fig. 31). Still another view is possible. We may say that $a + bi$ is represented geometrically by the *directed line* OP. Such a directed line is called a ***vector***.

From this point of view the geometric representation of a complex number is comparable to a force, or to a velocity, which is given when its magnitude, its direction, and its point of application are given.

Let OP_1 and OP_2 represent the complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i_1$ (Fig. 32). Construct the parallelogram OP_1PP_2,

Fig. 32.

with diagonal OP. Draw the ordinates M_1P_1, M_2P_2, and MP, and draw P_1N parallel to the axis of reals. From the equal triangles OM_2P_2 and P_1NP, $M_2P_2 = NP = b_2$ and $OM_2 = PN = a_2$. Consequently,

$$OM = OM_1 + M_1M = OM_1 + P_1N = a_1 + a_2,$$

and

$$MP = MN + NP = M_1P_1 + M_2P_2 = b_1 + b_2.$$

The coördinates of P are, therefore, $a_1 + a_2$ and $b_1 + b_2$, so that the point P, or the directed line OP represents the number $a_1 + a_2 + (b_1 + b_2)i$ which is the sum of the numbers $a_1 + b_1 i$

and $a_2 + b_2 i$. Therefore, *if two complex numbers be represented by vectors, their sum is the diagonal of the parallelogram constructed upon the two vectors as sides.*

It follows that geometric addition of complex numbers is precisely like the composition of forces or of velocities in physics.

EXERCISES

1. Construct each of the following sums by constructing the directed line for each complex number.

(*a*) $(1+i)+(2-i)$. (*c*) $(3-5i)+(2+7i)$.
(*b*) $(3-2i)+(-4+3i)$. (*d*) $(5+6i)+(2+3i)$.

2. Frame the rule for geometric addition in such a way that it will apply to the simpler case of addition of segments of a line.

3. Show geometrically that

$$|(a_1+b_1 i)+(a_2+b_2 i)| \leqq |a_1+b_1 i| + |a_2+b_2 i|,$$

and state the theorem expressed by the formula in words.

CHAPTER XI

POLYNOMIALS — EQUATIONS OF ANY DEGREE

91. Definitions. An algebraic expression of the form

$$(1) \qquad f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n,$$

where n is a positive integer, and the coefficients $a_0, a_1, \cdots a_n$, are constants, is called a ***rational integral function***, or, briefly, a ***polynomial*** in x.

A ***zero*** of the polynomial (1), that is, a number which when substituted for the variable x, causes the polynomial to vanish, is a ***root*** of the algebraic equation

$$(2) \qquad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0.$$

The coefficients $a_0, a_1, a_2, \cdots a_n$, may be assumed to be integers without loss of generality.

92. The Fundamental Theorem of Algebra. One of the prime objects of investigation in the study of polynomials relates to the number, the values, and the character of the zeros.

To this end the most important step is a fundamental theorem, which may be stated as follows.

FUNDAMENTAL THEOREM OF ALGEBRA. *Every algebraic equation has at least one root.**

This theorem, whose proof presents many difficulties, is here assumed to be true.

* The first demonstration of this theorem was published in 1799 by the German mathematician Gauss. Proofs will be found in such books as DICKSON, *Theory of Equations*, BURNSIDE AND PANTON, *Theory of Equations*, or WEBER, *Lehrbuch der Algebra*.

93. The Factor Theorem. *If r is a root of the equation $f(x)=0$, $x-r$ is a factor of the polynomial $f(x)$, and conversely.*

By the fundamental theorem, the equation $f(x)=0$ has at least one root r. Consequently, $f(r)=0$, either numerically, or identically, so that $f(x)\equiv f(x)-f(r)$. But

$$\begin{aligned} f(x)-f(r) &\equiv a_0x^n + a_1x^{n-1} + \cdots + a_n - (a_0r^n + a_1r^{n-1} + \cdots + a_n) \\ &\equiv a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots + a_{n-1}(x-r). \end{aligned}$$

Every one of the terms $a_0(x^n - r^n)$, $a_1(x^{n-1} - r^{n-1})$, $\cdots$, $a_{n-1}(x-r)$ is divisible by $x-r$. Consequently, $f(x)$ is divisible by $x-r$.

To prove that the converse is true, note that if $x-r$ is a factor of $f(x)$, then

$$f(x) \equiv (x-r)f_1(x), \tag{3}$$

where $f_1(x)$ is a polynomial of degree one less than that of $f(x)$. Consequently,

$$f(r) \equiv (r-r)f_1(r) \equiv 0.$$

The theorem is, therefore, proved.

Corollary. *A polynomial*

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

of degree n is the product of n linear factors and may be written in the form

$$\boldsymbol{f(x) \equiv a_0(x-r_1)(x-r_2)\cdots(x-r_n)}, \tag{4}$$

where $r_1, r_2, \cdots r_n$ are the roots of the equation $f(x)=0$.

The proof follows immediately, if we note that the polynomial $f_1(x)$ in equation (3) is divisible by $x-r_2$, where r_2 is a root of the equation $f_1(x)=0$. Then by the factor theorem,

$$f_1(x) \equiv (x-r_2)f_2(x),$$

where $f_2(x)$ is of degree one less than that of $f_1(x)$, or two less than $f(x)$. The process can be continued until the quotient becomes a constant. Moreover, for every division the first term of the quotient has a_0 as its coefficient.

94. The Number of Roots. THEOREM. *An algebraic equation of degree n has n, and only n, roots.*

In § 93 it was shown that

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n),$$

where $r_1, r_2, \cdots, r_n$ are the zeros of $f(x)$. Clearly, $f(x)$ vanishes when x is put equal to any one of the r's. If k of the linear factors $x - r$ are equal, we say that the equation has k equal roots. With this understanding there are at least n zeros of the function; that is, at least n roots of the equation $f(x) = 0$.

Moreover, if r' be any number different from every one of the r's, then

$$f(r') = a_0(r' - r_1)(r' - r_2) \cdots (r' - r_n) \neq 0,$$

since no one of the factors can be zero. Therefore $f(x)$ has only n zeros; that is to say, $f(x) = 0$ has only n roots.

EXERCISES

1. Is 2 a zero of $x^3 - 6x^2 + 11x - 6$? of $x^3 + 6x^2 + 11x - 6$?

2. Assuming that 3 is a zero of $x^3 - 6x^2 + 11x - 6$, find all the linear factors.

3. One root of the equation

$$x^3 - 6x^2 + 6x + 8 = 0,$$

is 4. Find the other two.

4. Prove by the factor theorem that $x^n - 1$ is always divisible by $x - 1$, and by $x^2 - 1$ when n is even.

5. Prove that $x^n + 1$ is divisible by $x + 1$ when n is odd, but not when n is even.

6. Prove that $yz(y - z) + zx(z - x) + xy(x - y)$ is divisible by $x - y$, $y - z$, and $z - x$.

95. The Graph of $f(x)$. The graph of $f(x)$ may be found by constructing a table of values of the function exactly as was done in Chapter V. Owing, however, to the greater number of turns in the graph of a polynomial of degree above 2 the student

will find it advisable to use shorter intervals between values of the independent variable.

Thus, if it be required to construct a graph of the function

$$f(x) \equiv x^3 - 9x^2 + 21x - 5,$$

the table of values to the nearest tenth, from $x = 0$ to $x = +6$, where the value of $f(x)$ is found for intervals of $\frac{1}{2}$, is as follows:

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
$f(x)$	-5	3.4	8	9.6	9	6.9	$+4$	1.1	-1	-1.6	0	4.6	13

When a smooth curve is drawn through the points given by the table, it is seen to have the form indicated in Fig. 33.

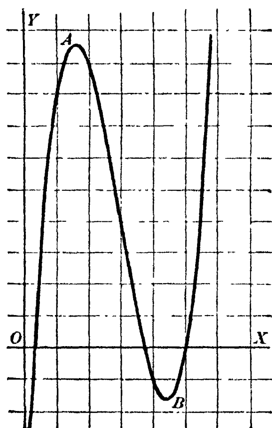


FIG. 33.

Not only does the graph in Fig. 33 give visual evidence of the fact that this cubic equation has three roots, but it tells us approximately what they are. From the figure the curve is seen to cross the x-axis in the vicinity of the points $x = .3$, $x = 3.7$, and $x = 5$. Moreover, the position and values of the maximum and minimum ordinates are approximately known.

When the roots are all real and are all known, an approximate idea of the form of the graph is easily obtained.

For example, suppose it is required to construct the graph of a function whose zeros are

$$-2, \ -1, \ 1, \ 2.$$

By the factor theorem the function is

$$f(x) \equiv (x+2)(x+1)(x-1)(x-2) \equiv x^4 - 5x^2 + 4.$$

For values of x less than -2, all four factors are negative. Therefore, $f(x)$ is positive, and consequently, the graph crosses from the positive to

the negative side of the x-axis at $x = -2$. It crosses the x-axis again in the points $x = -1$, $x = 1$, and $x = 2$.

The graph has the form indicated in Fig. 34.

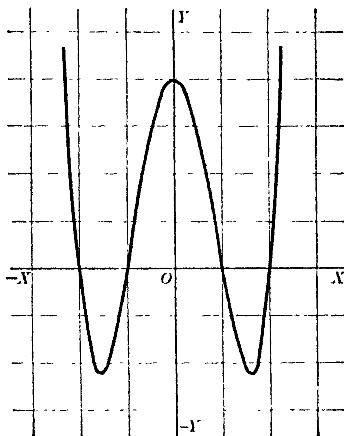


FIG. 34.

EXERCISES

1. Construct the graphs of each of the following functions, and locate approximately the real roots.

(a) $x^3 - 6x^2 + 6x + 8$.
(b) $x^3 - 5x^2 + 4x + 6$.
(c) $x^3 - 5x^2 + 5x + 3$.
(d) $x^3 - 6x^2 + 7x + 2$.
(e) $x^4 - 6x^3 + 6x^2 + 6x + 1$.
(f) $x^4 - 4x^3 + x^2 + 6x + 2$.

2. Construct roughly the graphs of each of the following expressions.

(a) $(x-1)(x+1)(x-2)$.
(b) $(x-2)(x-3)(x-4)(x-7)$.
(c) $(x^2 - 5x + 6)(x^2 - 12x + 35)$.
(d) $(x-2)^2(x-5)(x-7)$.

3. Prove that if $r_1, r_2, \cdots r_n$ are all real and a_0 is positive, the graph of

$$a_0(x - r_1)(x - r_2) \cdots (x - r_n)$$

is above the x-axis, for values of x which are less than the smallest r when n is even, and below the x-axis, when n is odd.

4. An equation of odd degree has at least one real root

96. The Approximate Values of the Real Roots. No simple method for determining the approximate values of the real roots exists.* If there is a single real root (or an odd number) between two given values of x, the fact is disclosed by means of the following theorem, the truth of which is almost self-evident.

If $f(x)$ be a rational integral function, and if for any two real values of x, $x = a$, and $x = b$, $f(a)$ and $f(b)$ have opposite signs, at least one real root of the equation $f(x) = 0$ lies between a and b.

For clearness suppose that $f(a)$ is negative and $f(b)$ is positive. Then, since $f(x)$ is continuous, as x passes from $x = a$ to $x = b$, $f(x)$ passes through all real values between $f(a)$ and $f(b)$. One of these values is zero. The corresponding value of x is therefore a root of the equation $f(x) = 0$.

From the geometric point of view the theorem states that if the graph is below the x-axis at $x = a$, and above it at $x = b$, it must cross the axis once, or an odd number of times, between $x = a$ and $x = b$.

If the real roots of an equation do not lie too close together, this theorem furnishes a convenient means of locating them between two consecutive integers.

For example, if

$$f(x) \equiv x^4 - 6x^3 + 5x^2 + 10x + 2,$$

the following table of values is easily computed:

x	-1	0	1	2	3	4	5
$f(x)$	4	2	12	10	-4	-6	52

By the theorem, one root lies between 2 and 3, and another between 4

* A complete solution of the problem of determining the number of real roots between any two values of x was published in 1835 by J. C. F. Sturm. For the demonstration of Sturm's theorem see DICKSON, *Theory of Equations*, or BURNSIDE AND PANTON, *Theory of Equations*.

and 5. Closer examination shows that $f(-.5)=-.9375$. Consequently, one of the two remaining roots lies between $x=-1$ and $x=-.5$, and the other between $x=-.5$ and $x=0$.

EXERCISES

1. Locate approximately the real roots of the following equations.

(a) $x^4-x^3-2x^2-3x-1=0.$

(b) $x^4-12x^3+36x^2-12x-5=0.$

(c) $x^4-5x^3+4x^2-3x-1=0.$

(d) $2x^5-13x^4+13x^3-x^2+14x-1=0.$

97. The Remainder Theorem. The formation of the table for a polynomial is greatly facilitated by an important theorem which may be stated as follows.

Remainder Theorem. *If a rational integral function of x be divided by $x-a$, the remainder is $f(a)$.*

Proof. Whatever value $f(a)$ may have,

$$f(x)\equiv f(x)-f(a)+f(a).$$

But

$$f(x)-f(a)\equiv a_0(x^n-a^n)+a_1(x^{n-1}-a^{n-1})+\cdots a_{n-1}(x-a),$$

or,

$$f(x)\equiv a_0(x^n-a^n)+a_1(x^{n-1}-a^{n-1})+\cdots+a_{n-1}(x-a)+f(a).$$

Every term on the right of the last identity, except $f(a)$, is divisible by $x-a$. Therefore,

$$\frac{f(x)}{x-a}\equiv Q(x)+\frac{f(a)}{x-a}, \tag{5}$$

where $Q(x)$, which is the quotient of $f(x)-f(a)$ by $x-a$, is a rational integral function of x. From (5) it follows that the remainder is $f(a)$.

If both sides of (5) be multiplied by $x-a$, the identity takes the form

$$f(x)\equiv(x-a)Q(x)+f(a). \tag{6}$$

By means of the remainder theorem the value of $f(a)$ may be found as the remainder after dividing $f(x)$ by $x - a$.

For example, if the function

$$f(x) \equiv x^4 - 3x^3 + 2x^2 - x + 13$$

be divided by $x - 2$, the quotient is

$$Q(x) = x^3 - x^2 - 1,$$

and the remainder is 11. Therefore $f(2) = 11$.

EXERCISES

1. Find the value of $f(3)$ when $f(x) = x^4 - 7x^3 + 8x^2 - 6x + 14$, and verify the result by actual substitution of 3 for x.

2. If $f(x) = 3x^3 - 5x^2 + 4x + 19$, what is $f(4)$? Verify the result.

3. Prove the factor theorem by means of the remainder theorem.

98. Synthetic Division. The work of finding $f(a)$ may be still further shortened by an abbreviated process for division called ***synthetic division.*** In order to explain this method of division, let $f(x)$ be of degree 4; then

$$f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4.$$

When $f(x)$ is divided by $x - a$, the quotient is of degree three with a constant remainder R. Therefore equation (6) of § 97 has the form

$$(7)\quad a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \equiv (x - a)(Ax^3 + Bx^2 + Cx + D) + R,$$

where A, B, C, D, and R are *undetermined coefficients.* When the multiplication indicated in (7) is carried out, the identity has the form

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$
$$\equiv Ax^4 + (B - aA)x^3 + (C - aB)x^2 + (D - aC)x + R - aD.$$

But if two polynomials are identically equal, the coefficients of like powers of the polynomials are equal. Therefore

$$a_0 = A,\ a_1 = B - aA,\ a_2 = C - aB,\ a_3 = D - aC,\ a_4 = R - aD;$$

or,

$$
(8) \qquad
\begin{aligned}
A &= a_0, \\
B &= a_1 + aA, \\
C &= a_2 + aB, \\
D &= a_3 + aC, \\
R &= a_4 + aD.
\end{aligned}
$$

The undetermined coefficients A, B, C, D, and R are therefore completely determined by equations (8).

The equations (8) may be written vertically instead of horizontally as in the following scheme:

$$
\begin{array}{ccccc}
a_0 & a_1 & a_2 & a_3 & a_4 \\
 & aA & aB & aC & aD \\
\hline
A & B & C & D & R
\end{array}
$$

This arrangement of the coefficients shows how the division may be performed mechanically. The examples will make the matter clear.

EXERCISES

1. By synthetic division find required values of the following functions.

(a) $f(2)$, when $f(x) \equiv x^3 - 7x^2 + 3x + 14$.
(b) $f(3)$, when $f(x) \equiv x^3 - 7x^2 + 3x + 14$.
(c) $f(4)$, when $f(x) \equiv x^3 - 7x^2 + 3x + 14$.
(d) $f(2)$, when $f(x) \equiv 3x^3 + 5x^2 - 9x + 14$.
(e) $f(.1)$, when $f(x) \equiv x^3 - 7x^2 + 3x + 14$.
(f) $f(-2)$, when $f(x) \equiv x^3 - 7x^2 + 3x + 14$.

2. Formulate a rule for dividing synthetically by $ax - b$

3. Divide synthetically $2x^3 - 3x^2 - 2x + 1$ by $2x - 3$.

4. Construct the graphs of each of the following functions after finding the values of the functions by synthetic division.

(a) $x^3 - 3x^2 + 7x - 7$.
(b) $2x^3 - 3x^2 + 4x - 12$.
(c) $5x^3 - 3x^2 + 4x - 1$.
(d) $4x^4 - 3x^3 + 4x^2 - 7x + 6$.
(e) $x^3 - 7$.
(f) $3x^4 - 4x^2 + 6$.

99. Rational Roots. By the factor theorem the equation whose roots are $r_1, r_2, r_3, \cdots r_n$ is

(9) $$(x-r_1)(x-r_2)\cdots(x-r_n)=0.$$

Suppose the roots are fractional and let

$$r_1=\frac{N_1}{D_1},\ r_2=\frac{N_2}{D_2},\ \cdots,\ r_n=\frac{N_n}{D_n},$$

where each fraction is in its lowest terms. The linear factors then become

$$\frac{D_1x-N_1}{D_1},\ \frac{D_2x-N_2}{D_2},\ \cdots,\ \frac{D_nx-N_n}{D_n}$$

and when the equation (9) is multiplied through by the product $D_1D_2\cdots D_n$, it takes the form

(10) $$(D_1x-N_1)(D_2x-N_2)(D_3x-N_3)\cdots(D_nx-N_n)=0.$$

From the form (10),

(11) $$\boldsymbol{a_0=D_1D_2D_3\cdots D_n}, \text{ and } \boldsymbol{a_n=(-1)^nN_1N_2\cdots N_n}.$$

It is clear from equations (11) that *the denominator of any rational root is a divisor of a_0, and the numerator is a divisor of a_n.* It follows that the rational roots may be found by trial, since the number of integral divisors of a_0 and a_n is finite.

EXERCISES

1. Find the roots of the equation

$$4x^4+4x^3-5x^2-9x-9=0.$$

SOLUTION. Since the denominator of any rational root must be a divisor of 4 and the numerator of 9, the rational roots are to be found in the set of numbers,

$$\pm 1,\ \pm 3,\ \pm 9,\ \pm \tfrac{1}{2},\ \pm \tfrac{1}{4},\ \pm \tfrac{3}{2},\ \pm \tfrac{3}{4},\ \pm \tfrac{9}{2},\ \pm \tfrac{9}{4}.$$

Actual trial shows that $\frac{3}{2}$ and $-\frac{3}{2}$ are roots. The remaining roots may be found from the equation,

$$\frac{4x^4+4x^3-5x^2-9x-9}{4x^2-9}\equiv x^2+x+1=0.$$

2. Find the roots of each of the following equations.

(*a*) $x^3-x^2-4x-6=0.$ (*c*) $6x^4-35x^3+68x^2-40x+7=0.$

(*b*) $2x^3-x^2-14x+10=0.$ (*d*) $12x^4-65x^3+86x^2-41x+6=0.$

100. The Transformation of Equations. The operation of changing one equation into another whose roots bear a definite relation to the roots of the first, is called a ***transformation.*** The most important transformation is one that changes the equation into another whose roots are greater or less by a fixed number than the roots of the original equation.

PROBLEM I. *To diminish the roots of an equation by a fixed number.*

Suppose it be required to diminish the roots of the equation

$$f(x) \equiv a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0$$

by the fixed number h. If x be replaced everywhere by $x' + h$, the new equation is

$$(12) \quad f(x' + h) \equiv a_0(x' + h - r_1)(x' + h - r_2) \cdots (x' + h - r_n) = 0.$$

The roots of the equation (12), obtained by setting the separate linear factors equal to zero, are

$$(13) \qquad x'_1 = r_1 - h, \quad x'_2 = r_2 - h, \quad \cdots, \quad x'_n = r_n - h.$$

That is to say, each root of the new equation is equal to the corresponding root of the old equation diminished by h. The required transformation is therefore accomplished by writing everywhere $x' + h$ for x, or, as it is ordinarily expressed, by the *substitution*,

$$(14) \qquad x = x' + h.$$

If h is a negative number, the transformation (14) will increase all the roots by h.

Geometrically, this transformation changes the position of the graph of $f(x)$ by moving it bodily from right to left, or from left to right, according as h is positive or negative. It is often used to transform an equation into another which lacks a specified term. (See Ex. 4 below.)

PROBLEM II. *To change the signs of the roots of an equation.* Another important transformation is one which changes the equation into another whose roots are the negatives of the roots of the original equation. It is accomplished by the substitution $x = -x'$. The proof is left to the student.

EXERCISES

1. Transform the equation $f(x) \equiv x^3 - 3x^2 + 2x - 3 = 0$ into another whose roots are less by 2.

SOLUTION. The required transformation is $x = x' + 2$ and, consequently, the required equation is

$$f(x' + 2) \equiv (x' + 2)^3 - 3(x' + 2)^2 + 2(x' + 2) - 3 = 0.$$

When the operations indicated are carried out and the accents dropped, the equation reduces to $x^3 + 3x^2 + 2x - 3 = 0$.

2. Transform the equation $x^3 - 3x^2 + 14x - 1 = 0$ into another whose roots are greater by 2.

3. Transform the equation $x^4 - 9x^3 + 9x^2 + 8x + 18 = 0$ into another having a root between 0 and 1.

[HINT. By the theorem of § 96 it is found that one root of the given equation lies between 2 and 3.]

4. Transform the equation $x^3 - 3x^2 + 14x - 1 = 0$ into another in which the coefficient of x^2 is zero.

[HINT. Let $x = x' + h$; then $f(x' + h) = (x' + h)^3 - 3(x' + h)^2 + 14(x' + h) - 1 = x'^3 + 3hx'^2 - 3x'^2 +$ terms of lower degree The coefficient of x'^2 in the new equation is $3h - 3$ and this coefficient will be zero for $h = 1$.]

5. Transform the equation $x^4 - 9x^3 + 9x^2 + 8x + 18 = 0$ into another which lacks the term in x^3.

6. Find a method by which an equation may be transformed into another whose roots are the roots of the original equation each multiplied by a given number m.

7. The roots of the equation $12x^3 - 40x^2 + 41x - 12 = 0$ are rational, but not integral. Transform the equation into another whose roots are integers.

[HINT. The coefficient of x^3 is a multiple of the denominators of all the roots. Use the result of Ex. 6.]

8. Transform the equation $x^3 + 4x^2 + 3x - 14 = 0$ into another which lacks the term in x^2.

9. Prove that any equation of degree n can be transformed into another which lacks the term of degree $n - 1$. Give the substitution when the equation is written in the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

10. Find the condition that the cubic equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0$$

can be transformed into another which lacks the term of first degree, by a *real rational transformation*.

11. Solve the quadratic equation $a_0x^2 + a_0x + a_2 = 0$, by first transforming it to the form $Ax^2 + B = 0$.

12. Find the approximate values of the roots of the following equations by first removing the second term and then using the graphical methods of §§ 83 and 84.

(*a*) $x^3 - 3x^2 + 14x - 1 = 0.$
(*b*) $x^4 - 9x^3 + 9x^2 + 8x + 18 = 0.$
(*c*) $x^4 - 6x^3 + 8x^2 + 4x - 4 = 0.$

101. Horner's Method for the Approximate Determination of Real Roots. When the roots of an equation of degree greater than 2 are irrational, it is usually impossible to find usable values for the real roots except by methods of approximation.* The most useful elementary method of approximation is known as Horner's method.

Horner's method consists essentially in locating a root between two integers, and then diminishing this root by successive transformations until an equation is reached whose root differs from zero by an amount as small as we please. The details of the method will be made clear by an example.

* Equations of the third and fourth degrees may be solved algebraically by well-known methods, though the forms of the solutions are such that they are not easily available for practical purposes. These solutions may be found in works on advanced algebra. It was proven by Abel in 1826 that the algebraic solution of the general equation of degree greater than four is impossible.

First Transformation. Let it be required to find the approximate value of a root of the equation

(15) $$f(x) = x^4 + x^3 - 11x^2 + 12x - 10 = 0.$$

By theorem of § 96, it is easily shown that one root lies between $x = 2$ and $x = 3$. If, therefore, this equation be transformed by the substitution $x' = x + 2$, the corresponding root of the new equation will lie between 0 and 1. Since the *form* of the new equation is known, we may write

$$f(x' + 2) \equiv (x' + 2)^4 + (x' + 2)^3 - 11(x' + 2)^2 + 12(x' + 2) - 10$$
$$\equiv A_0x'^4 + A_1x'^3 + A_2x'^2 + A_3x + A_4.$$

But since

$$x = x' + 2, \quad x' = x - 2.$$

Therefore if x' be replaced by $x - 2$, the resulting function will be identical with the original $f(x)$. That is,

$$f(x) \equiv x^4 + x^3 - 11x^2 + 12x - 10$$
$$\equiv A_0(x-2)^4 + A_1(x-2)^3 + A_2(x-2)^2 + A_3(x-2) + A_4.$$

Clearly, the remainder obtained by dividing the right member of this identity by $x - 2$ is A_4. But the remainder after dividing the left member of the identity by $x - 2$ is -6. Therefore, $A_4 = -6$.

The integral part of the quotient obtained by dividing the left member is

$$Q_1(x) \equiv x^3 + 3x^2 - 5x + 2.$$

Consequently,

$$Q_1(x) \equiv x^3 + 3x^2 - 5x + 2$$
$$\equiv A_0(x-2)^3 + A_1(x-2)^2 + A_2(x-2) + A_3.$$

The coefficient A_3 may be found in exactly the same way that A_4 was found, namely, by dividing both sides of the new identity by $x - 2$. The process can be continued in this way until all the coefficients are found.

The successive quotients and remainders may be found by synthetic division and the work may be arranged as follows:

$$\begin{array}{llllll}
1 & 1 & -11 & 12 & -10 & (x-2 \\
 & 2 & 6 & -10 & 4 & \\
\hline
1 & 3 & -5 & 2 & -6 = A_4, & Q_1(x) \equiv x^3 + 3x^2 - 5x + 2. \\
 & 2 & 10 & 10 & & \\
\hline
1 & 5 & 5 & 12 = A_3, & & Q_2(x) \equiv x^2 + 5x + 5. \\
 & 2 & 14 & & & \\
\hline
1 & 7 & 19 = A_2, & & & Q_3(x) \equiv x + 7. \\
 & 2 & & & & \\
\hline
1 & 9 = A_1, & & & & Q_4(x) \equiv 1. \\
\hline
1 = A_0 & & & & &
\end{array}$$

The new equation is, therefore,

$$x^4 + 9x^3 + 19x^2 + 12x - 6 = 0. \tag{16}$$

SECOND TRANSFORMATION. Since a root of equation (15) lies between 2 and 3, the corresponding root of (16) must lie between 0 and 1. Closer examination shows that it lies between .3 and .4. If the roots of (16) be diminished by .3, the corresponding root of the new equation will lie between 0 and .1. The work is arranged as follows:

$$\begin{array}{llllll}
1 & 9 & 19 & 12 & -6 & (x-.3 \\
 & .3 & 2.79 & 6.537 & 5.5611 & \\
\hline
1 & 9.3 & 21.79 & 18.537 & -.4389 = A'_4, & Q'_1(x) \equiv x^3 + 9.3x^2 + 21.79x + 18.537. \\
 & .3 & 2.88 & 7.401 & & \\
\hline
1 & 9.6 & 24.67 & 25.938 = A'_3, & & Q'_2(x) \equiv x^2 + 9.6x + 24.67. \\
 & .3 & 2.97 & & & \\
\hline
1 & 9.9 & 27.64 = A'_2, & & & Q'_3(x) \equiv x + 9.9. \\
 & .3 & & & & \\
\hline
1 & 10.2 = A'_1, & & & & Q'_4(x) \equiv 1. \\
\hline
1 = A'_0 & & & & &
\end{array}$$

The new equation is

$$x^4 + 10.2x^3 + 27.64x^2 + 25.938x - .4389 = 0. \tag{17}$$

THIRD TRANSFORMATION. Since the root of equation (17) lies between 0 and .1, the sum of the first three terms will be very small, and the root will be found approximately by neglecting the higher powers of x and using the linear equation

$$25.938\,x - .4389 = 0.$$

Clearly, the root of this equation lies between .01 and .02, so that the third transformation must be arranged to diminish the root by .01. By using $x - .01$ as a divisor, equation (17) is transformed into

$$(18)\quad x^4 + 10.24\,x^3 + 27.9466\,x^2 + 26.493864\,x - 0.17674579 = 0.$$

Using the last two terms of equation (18), we find the root is approximately .006.

By transforming (18) we should find a fifth equation, one of whose roots lies between 0 and .001. In this way the work may be carried on until an equation is obtained with one root as near to zero as we please.

Collecting results obtained thus far, we find that the root which originally lay between 2 and 3, has been diminished by 2.31, and that it now lies between 0 and .001. It is evident, therefore, that 2.31 is the approximate value to the second decimal place.

The process of finding the next figure of the root by solving the linear equation obtained by discarding all terms of degree higher than 2, is frequently called the method of *trial divisors*. It is evident that the accuracy of the trial divisor found in this way, increases as the work proceeds. Indeed, the next *two* figures of the root could have been obtained from the equation

$$25.938\,x - .4389 = 0.$$

The approximate value of the root of the equation (15) to the fourth decimal place is 2.3166.

EXERCISES

1. Find the real roots to the third decimal place.

(*a*) $x^4 - 12x^3 + 36x^2 - 12x - 5 = 0$.

(*b*) $x^4 - 10x^2 + 1 = 0$.

(*c*) $x^4 - 5x^3 + 4x^2 - 3x - 1 = 0$.

(*d*) $2x^5 - 13x^4 + 13x^3 - x^2 + 14x - 12 = 0$.

2. Find the positive roots of the equation

$$x^3 + 12x^2 - 6008x + 179200 = 0.$$

[Hint. To locate the roots, try 10, 20, 30, ⋯.]

3. Find the roots of the equation

$$x^3 - 1882x^2 + 10695x - 201474 = 0.$$

4. When one end of a beam of length l, and carrying a uniform load, is "built in" to a wall and the other rests upon a fulcrum, the distance x from the fulcrum to the point of maximum deflection is given by the equation $8x^3 - 9lx^2 + l^3 = 0$.

Find x for a beam 10 feet long.

[Merriam, *Textbook on Mechanics of Materials*, p. 153.]

5. Find the 5th root of 349 to 4 places of decimals by applying Horner's method to the solution of the equation $x^5 - 349 = 0$.

6. If the slope made by a line with the x-axis is a, the slope x of a line whose angle with the x-axis is one third as great is given by the equation $x^3 - 3ax^2 - 3x + a = 0$. What is the smaller slope when the greater is 1? When the greater is 2?

7. A man invests \$100 a year for 5 years in a savings association and at the end of the period his stock is worth \$560.25. The formula for computing the amount saved in n years is

$$S = P\frac{(1+i)^n - 1}{i},$$

where P is the amount paid in each year and i is the rate. What rate of interest does he receive?

8. The volume v of a spherical segment of one base is

$$v = \pi\frac{(r-x)^2(x+2r)}{3},$$

where r is the radius of the sphere, and x the distance of the base from the center. What is x when the segment is one half a hemisphere and the radius is 3?

9. A rectangular beam 10 inches wide is used to brace a rectangular

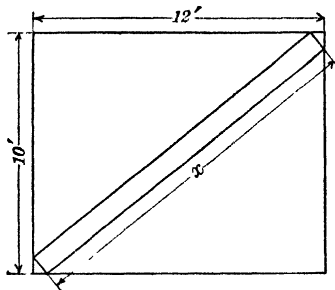


Fig. 35.

frame whose inner dimensions are 10 by 12 feet. (See Fig. 35.) What is the length of the beam exclusive of the tenon?

10. Bryan's equation for lateral stability of an aëroplane in horizontal steady motion, with coefficients determined by Bairstow for a machine of given dimensions, is

$$\lambda^4 + 9.31\,\lambda^3 + 9.81\,\lambda^2 + 10.15\,\lambda - 0.161 = 0.$$

Find one positive and one negative root to 4 decimal places.

[Technical Report, British Advisory Committee for Aëronautics.]

11. Bryan and Bairstow's equation for lateral stability of an aëroplane in downward flight 1 — 6 with propeller cut off, is

$$\lambda^4 + 9.31\,\lambda^3 + 9.81\,\lambda^2 + 10.25\,\lambda + 0.467 = 0.$$

Find two real roots to 4 decimal places. [*Ibid.*]

12. The theory of symmetrically reinforced concrete beams depends upon the equation

$$K^3 - 3\left(\frac{1}{2} - \frac{e}{h}\right) K^2 + 12\,np\frac{c}{h}K = 6\,np\left[\frac{e}{h} + 2\left(\frac{a}{h}\right)^2\right].$$

Find K to the second decimal place when

$$e/h = 0.6, \quad a/h = .04, \quad p = .01, \quad n = 15.$$

[Turneaure and Maurer, *Principles of Reinforced Concrete Construction*, 1912, p. 105.]

13. If, in using Horner's method, a trial divisor should be too large, how could the fact be detected?

102. Relations between the Roots and the Coefficients. To find the fundamental relations between the roots and the coefficients, the equation

$$f(x)=0$$

may first be reduced to the form

$$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=0, \tag{19}$$

by dividing both sides by a_0 and then writing

$$\frac{a_1}{a_0}=p_1,\ \frac{a_2}{a_0}=p_2,\ \cdots,\ \frac{a_n}{a_0}=p_n. \tag{20}$$

For clearness let $f(x)$ be of degree 4. By the factor theorem, it is evident that

$$x^4+p_1x^3+p_2x^2+p_3x+p_4\equiv(x-r_1)(x-r_2)(x-r_3)(x-r_4).$$

When the indicated multiplications are performed, the last identity takes the form,

$$\begin{aligned}x^4+p_1x^3+p_2x^2+p_3x+p_4\equiv x^4&-(r_1+r_2+r_3+r_4)x^3\\&+(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)x^2\\&-(r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)x+(r_1r_2r_3r_4).\end{aligned} \tag{21}$$

An identity similar to (21) holds for every equation which has the form (19), whatever its degree may be. In general, p_i is the sum of all possible products that can be found by taking the roots i at a time with the sign factor $(-1)^i$. We may write therefore

$$\begin{cases}p_1=-(r_1+r_2+\cdots+r_n)\equiv-\Sigma r_1,\\ p_2=(r_1r_2+r_1r_3\cdots+r_2r_3+\cdots+r_{n-1}r_n)\equiv\Sigma r_1r_2,\\ p_3=-(r_1r_2r_3+r_1r_2r_4+\cdots+r_{n-2}r_{n-1}r_n)\\ \quad\equiv-\Sigma r_1r_2r_3\ \ldots\ldots\ldots\ldots\\ p_n=(-1)^nr_1r_2r_3\cdots r_n.\end{cases} \tag{22}$$

The expressions Σr_1, $\Sigma r_1r_2\cdots$ which are abbreviations for the sums of the products, one at a time, two at a time, and so on are defined by the equations in which they occur.

EXERCISES

1. Give the sums of the products of the roots, one at a time, two at a time, and so on for each of the following equations.

(*a*) $x^5 - 5x^4 + 6x^3 + 7x^2 - 3x + 14 = 0$.

(*b*) $3x^4 + 7x^3 - 18x^2 + 14x - 19 = 0$.

(*c*) $x^{10} - 8x^5 + 17 = 0$.

(*d*) $x^{10} - 1 = 0$.

2. Write out without multiplying the linear factors $x - r$, the equations whose roots are as follows.

(*a*) 1, 2, 3, 4.

(*b*) 3, -2, 4, -5.

(*c*) 2, $2 + \sqrt{3}$, $2 - \sqrt{3}$.

(*d*) $2 + 3i$, $2 - 3i$, $4 + i$, $4 - i$.

103. The Character of the Roots. THEOREM. *The imaginary roots of an equation with real coefficients occur in pairs of conjugates.*

The theorem asserts that if $a + bi$ is a root, $a - bi$ is also a root. Consequently,

$$[x - (a + bi)][x - (a - bi)] = (x - a)^2 + b^2$$

must be a factor of $f(x)$.

PROOF. To prove this fact, we may examine the remainder after dividing $f(x)$ by $(x - a)^2 + b^2$. Clearly, the remainder cannot be of degree higher than 1. If it be denoted by $Sx + T$,

$$f(x) \equiv [(x - a)^2 + b^2]Q(x) + Sx + T, \tag{23}$$

where $Q(x)$ denotes the integral part of the quotient, and S and T are real numbers.

By hypothesis, $f(a + bi) \equiv 0$, that is,

$$[(a + bi - a)^2 + b^2]Q(a + bi) + S(a + bi) + T = 0.$$

Since $(a + bi - a)^2 + b^2 \equiv 0$, we must have

$$S(a + bi) + T \equiv Sa + T + Sbi = 0.$$

But by § 87 if a complex number is equal to zero, the real and the imaginary parts are separately equal to zero. That is to say,

$$Sa + T = 0, \quad \text{and} \quad S = 0.$$

But if $S = 0$, $T = 0$ also, so that the remainder $Sx + T$ is zero, and the division is exact. The theorem is therefore proved.

COROLLARY I. *An equation of odd degree with real coefficients has at least one real root.*

COROLLARY II. *Any rational integral function with real coefficients may be broken up into real factors, all of which are of the first, or of the second, degree.*

104. The Graph of $f(x)$ when Some Factors are Imaginary. Since imaginary roots occur in pairs of conjugates, $f(x)$ will have a quadratic factor $(x-a)^2+b^2$ corresponding to every such pair. Moreover, according to § 66, the sign of this quadratic factor never changes. Every such quadratic factor therefore diminishes the number of intersections with the x-axis by two. Usually, but not always, the graph has an *elbow* corresponding to each pair of imaginary roots, as in Fig. 36. Geometrically, these considerations become almost self-evident if we imagine the x-axis to be lowered (or the curve raised) in Fig. 33 until the point B lies above the axis. The curve in Fig. 36 is identical with that of Fig. 33, but each of its points are three units higher than the corresponding point in Fig. 33.

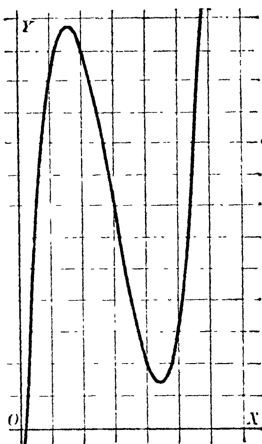


FIG. 36.

EXERCISES

1. Sketch roughly the graph of each of the following functions.

(*a*) $f(x)=(x-1)(x-2)(x^2+x+1)$.
(*b*) $f(x)=(x-3)(x^2-4x+13)$.
(*c*) $f(x)=(x-3)^2(x^2-4x+13)$.
(*d*) $f(x)=(x^2+x+1)(x^2-x+1)$.
(*e*) $f(x)=(x^2+x+1)(x^2-x+1)(x^2-4x+13)$.

105. The Number of Real Roots. Descartes's Rule of Signs. Sturm's theorem, to which reference was made in § 96, is the only method known for the determination of the number of real roots of a numerical equation. There is, however, a simpler method due to Descartes which, in every case, gives an upper bound to the number.

Any change in sign between two consecutive terms of $f(x)$ is called a *variation.* Thus $x^4 + 3x^3 - 2x^2 - x + 13$ contains two variations, one from $+$ to $-$ and the other from $-$ to $+$.

DESCARTES'S RULE. *An equation $f(x) = 0$ cannot have more real positive roots than there are variations of sign in $f(x)$, nor more real negative roots than there are variations of sign in $f(-x)$.*

Suppose the product of all quadratic factors corresponding to pairs of conjugate roots is a function for which the sequence of signs is

$$+ + - + + - - + + +.$$

To introduce another root r, which is real and positive, it is necessary to multiply the function by $x - r$, for which the sequence of signs is $+ -$. The multiplication of signs may be carried out as follows:

$$\begin{array}{ccccccccccc}
+ & + & - & + & + & - & - & + & + & + & \\
+ & - & & & & & & & & & \\
\hline
+ & + & - & + & + & - & - & + & + & + & \\
 & - & - & + & - & - & + & + & - & - & - \\
\hline
+ & \pm & - & + & \pm & - & \pm & + & \pm & \pm & -
\end{array}$$

In the product the double sign indicates that the sign is in doubt. It is not difficult to see that the sign in the last line which stands directly underneath the *second* sign of any variation, will be the same both in the multiplicand and the product. It follows, therefore, that if we consider only those signs up to and including the second sign of the last variation, the number of variations in that part of the product standing

directly underneath will be at least as great. But one variation is introduced in the remaining terms, because the last sign of the product is different from the last sign of the multiplicand. Therefore, at least one variation has been introduced.

The conclusion just reached holds whatever the sequence of signs in the multiplicand may be. Since the introduction of each real positive root increases the number of variations by at least one, it follows that the equation cannot have more real positive roots than it has variations of sign.

To prove the second part of the rule it is only necessary to note that the roots of $f(-x) = 0$ are the negatives of the roots of $f(x) = 0$. (§ 100.)

Example. The equation

$$f(x) \equiv x^6 - 3x^5 - 5x^3 - x + 1 = 0$$

cannot have more than two positive roots; and since

$$f(-x) \equiv x^6 + 3x^5 + 5x^3 + x + 1 = 0,$$

it cannot have any negative root. It must, therefore, have at least four imaginary roots.

EXERCISES

1. Prove that the equation $x^5 - 1 = 0$ has only one real root.

2. Prove that the equation $x^4 - 3x^3 + 1 = 0$ has at least two complex roots.

3. Prove that the equation $x^n - 1 = 0$ has $n - 1$ complex roots when n is odd and $n - 2$ when n is even.

4. Prove that $x^n + 1 = 0$ has no real root when n is even, and one, when n is odd.

5. What is the number of real roots of the equation

$$x^5 + x^4 + x^3 + x^2 + x + 1 = 0\,?$$

[Hint. Increase the degree by introducing the root 1.]

6. What is the number of real roots of the equation

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = ?$$

7. Generalize the results in Exs. (5) and (6).

8. The formula for the amount of an annuity is

$$S = a\frac{(1+i)^n - 1}{i},$$

and when S, a, and n are known the formula becomes an equation for the determination of the rate of interest. By substituting z for $1 + i$, prove that the equation has only one root that can have any significance in financial problems.

9. Prove that if the coefficients of $f(x)$ are alternately positive and negative, the equation $f(x) = 0$ has no negative root.

106. The Highest Common Factor of Two Polynomials. Let us consider two polynomials

$$f(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_m,$$
$$\phi(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n,$$

whose degrees in x are m and n, respectively. Let us suppose that $m \geqq n$.

Only in exceptional cases can these polynomials be factored. The method for finding the highest common factor given in § 20, therefore, fails, and some other must be sought.

If $f(x)$ be divided by $\phi(x)$, the quotient will be made up of two parts, one an integral part $Q_1(x)$, which is a rational integral function of x, and the other, a fractional part whose numerator $R_1(x)$ is a rational integral function whose degree is an integer less than the degree of the divisor $\phi(x)$. The result may be expressed in either of the two forms

$$\frac{f(x)}{\phi(x)} \equiv Q_1(x) + \frac{R_1(x)}{\phi(x)}, \tag{24}$$

or

$$f(x) \equiv Q_1(x)\phi(x) + R_1(x).$$

Similarly, if $\phi(x)$ be divided by $R_1(x)$, the remainder $R_2(x)$ will be of degree less than that of $R_1(x)$. This process may be carried on until a remainder is reached which is either zero or a constant. The results may be expressed in the following identities, which taken together are called the ***Euclidean algorithm:***

$$(25)\qquad \begin{cases} f(x) \equiv Q_1(x)\phi(x) + R_1(x), \\ \phi(x) \equiv Q_2(x)R_1(x) + R_2(x), \\ R_1(x) \equiv Q_3(x)R_2(x) + R_3(x), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ R_{l-3}(x) \equiv Q_{l-1}(x)R_{l-2}(x) + R_{l-1}(x), \\ R_{l-2}(x) \equiv Q_l(x)R_{l-1}(x) + R_l(x). \end{cases}$$

From the first identity of the Euclidean algorithm, it is clear that any divisor of $f(x)$ and $\phi(x)$ is a divisor of $R_1(x)$ and from the second, that this same divisor is a divisor of $R_2(x)$, and so on. It is clear, therefore, that when $R_l(x) = 0$, $R_{l-1}(x)$ is the *highest common factor*, and that when $R_l(x) \neq 0$, no highest factor, apart from a constant, exists.

It is important to notice that Euclid's process for finding the highest common factor involves only operations leading to *rational* results. Furthermore, the result will not be affected, if at any stage of the work the dividend be multiplied or divided by any constant.

EXERCISES

1. Find the highest common factor for each of the following pairs of polynomials.

(*a*) $x^4 - 3x^3 + x^2 + 4$ and $4x^3 - 9x^2 + 2x$.

(*b*) $x^3 - 1$ and $x^3 - 6x + 11x - 6$.

(*c*) $x^3 - 5x^2 + 17x - 13 = 0$ and $x^4 - 3x^3 + 10x^2 + 9x + 13$.

(*d*) $x^4 - 16$ and $x^4 + x^3 + 5x^2 + 4x + 4$.

(*e*) $x^4 - 16$ and $x^3 - 6x^2 + 11x - 6$.

(*f*) $x^4 - 16$ and $x^3 - 8x^2 + 17x - 12$.

107. Roots Common to Two Equations. Multiple Roots. Let $f(x) = 0$ and $\phi(x) = 0$ be two equations of degrees m and n. Then,

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_m),$$
$$\phi(x) = b_0(x - r'_1)(x - r'_2) \cdots (x - r'_n),$$

where $r_1, r_2, \cdots, r_m$ are the roots of $f(x) = 0$, and $r'_1, r'_2, \cdots r'_n$ are the roots of $\phi(x) = 0$.

If a certain number of roots are common to the two equations, the corresponding linear factors are common to $f(x)$ and $\phi(x)$. It follows, therefore, that the highest common factor of $f(x)$ and $\phi(x)$ will contain those, and only those, linear factors which correspond to common roots. If $g(x)$ be the highest common factor, the common roots may be found by solving the equation $g(x) = 0$.

An important application of the foregoing paragraph is the determination of multiple roots of an equation, that is, roots which correspond to linear factors that are repeated.

In the calculus, it is shown that a multiple root of an equation $f(x) = 0$ is common to the two equations

$$(26) \quad f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$
$$(27) \quad f'(x) \equiv na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + a_{n-1} = 0.$$

The function $f'(x)$ is called the ***derivative*** of $f(x)$. It is formed from $f(x)$ by multiplying the coefficient of each term of $f(x)$ by the exponent of x and then diminishing that exponent by 1.

EXAMPLE. If $\quad f(x) \equiv x^3 - 5x^2 + 8x - 4,$

then $\quad f'(x) \equiv 3x^2 - 10x + 8.$

The highest common factor of $f(x)$ and $f'(x)$ is $x - 2$. Therefore $x = 2$ is a multiple root of $f(x) = 0$. Since $(x - 2)^2$ is a factor of $f(x)$, the remaining linear factor is easily found to be $x - 1$. The roots of the equation are therefore 2, 2 and 1.

EXERCISES

1. Find the roots common to the following pairs of equations.

(*a*) $x^4 - 3x^3 + x^2 + 4 = 0$ and $4x^3 - 9x^2 + 2x = 0$.

(*b*) $x^3 - 5x^2 + 17x - 13 = 0$ and $x^4 - 3x^3 + 10x^2 + 9x + 13 = 0$.

(*c*) $x^4 - 16 = 0$ and $x^4 + x^3 + 5x^2 + 4x + 4 = 0$.

2. Examine the following equations for multiple roots.

(*a*) $x^3 - 9x^2 + 24x - 16 = 0$.

(*b*) $x^4 - 2x^3 - 11x^2 + 24x - 12 = 0$.

(*c*) $x^5 - x^4 - 24x^3 + 24x^2 + 144x - 144 = 0$.

(*d*) $x^6 - 5x^5 + 8x^4 - 5x^3 + 5x^2 - 8x + 4 = 0$.

3. Prove that an equation of the form $x^n \pm a = 0$ cannot have a multiple root.

CHAPTER XII

DETERMINANTS AND LINEAR EQUATIONS

108. Definitions. Determinants of orders two and three were defined in §§ 70 and 73. A ***determinant of order n*** is a square array of n^2 elements, which is interpreted to mean *the algebraic sum of all possible products that can be found by taking one, and only one, element from each row and each column, with the signs of the terms determined according to the rule of signs explained below.* A determinant of fourth order is written in the form,

$$(1) \qquad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

or, briefly,

$$(a_1b_2c_3d_4).$$

Any term of this determinant will be made up of factors a, b, c, d, with subscripts all different, since, if two letters were alike, the product would contain two elements from the same column, and if two subscripts were alike, two elements from the same row. When the letters are written in the normal order, that is, the order in which they occur in the alphabet, the subscripts will occur in every possible order.

An ***inversion*** from the order of the natural scale occurs whenever a digit stands before another that, in the natural scale, precedes it. Thus, in the term $a_1b_4c_3d_2$, 4 stands before 3 and before 2, and 3 stands before 2. There are then three inversions in the term.

THE RULE OF SIGNS. *When the letters are arranged in the normal order, the sign of a term is plus or minus according as the number of inversions of subscripts is even or odd.*

By the rule, the sign of the term $a_1b_4c_3d_2$ is minus, while the sign of the term $a_1b_4c_2d_3$ is plus.

The algebraic sum of the terms written out according to the definition, is called the ***expansion*** of the determinant.

Since the subscripts will occur in every possible order, the number of terms is equal to the number of possible *arrangements* or *permutations*, of the subscripts. In a later section it will be shown that this number is $n!$, where $n!$ is defined to be the product of the n integers 1, 2, 3, ⋯, n. Thus, the number of terms in the expansion of a determinant of order 3, is 6; of order 4, is 24; of order 5, is 120.

EXERCISES

1. Write out the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

2. Find the value of the determinant

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 2 \\ 3 & 2 & 5 \end{vmatrix},$$

by writing out the terms according to the definition.

3. Find the value of the determinant

$$\begin{vmatrix} 2 & 3 & 5 & 7 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 5 & 8 \\ 4 & 7 & 3 & 6 \end{vmatrix}.$$

109. General Properties of Determinants. A few of the more important properties of determinants are given by the following theorems.

THEOREM 1. *The value of a determinant is not changed if the rows and columns be interchanged, leaving the relative position of the elements in the row (or column) unchanged.*

In symbols the theorem states that

$$(2) \qquad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

In the second determinant the subscripts are *row marks*, and the letters, *column marks*, while the reverse is true for the first. Any term, as $a_2b_3c_1d_4$, is a term of the second determinant, and, except possibly for a sign, of the first also. When this term is written in the form $c_1a_2b_3c_4$, the number of inversions of letters (column marks for the first determinant) is the same as the number of inversions of subscripts (row marks for the second determinant) in the form $a_2b_3c_1d_4$; so that the sign is the same for both determinants. That this will be true, in general, is seen from the fact that every step taken to bring the subscripts of a term like $a_2b_3c_1d_4$ into their natural order, introduces an equal number of inversions of the letters.

THEOREM 2. *If two rows (or two columns) of a determinant be interchanged, the sign of the determinant is changed but its numerical value is unchanged.*

Suppose, first, that two adjacent rows, say the second and third, be interchanged. The subscripts 2 and 3 occur in every term of the new, as well as of the old determinant, and in the new determinant the order for the purpose of sign determination is 1, 3, 2, 4. Wherever 3 occurs before 2, there is an inversion for the original, but not for the new determinant.

and wherever 2 occurs before 3 it is not an inversion for the original, but is for the new determinant. Consequently, any given term considered as a term of the new determinant has exactly one inversion more, or one fewer than it has when considered as a term of the original determinant. Therefore, when two adjacent rows (or columns) are interchanged the sign of the determinant, and the sign only, is changed.

If two rows with k rows between are interchanged, one row can be made to stand adjacent to the other by k interchanges of adjacent rows. Then the second may be moved to the position occupied by the first by $k+1$ interchanges of adjacent rows. The interchange of two non-adjacent rows is therefore effected by $2k+1$ interchanges of adjacent rows. The $2k+1$ changes of sign leaves the determinant with sign changed.

COROLLARY. *A determinant in which the elements of a row (or column) are equal to corresponding elements of another row (or column), vanishes identically.*

For if the identical rows (or columns) be interchanged the determinant is not changed, though the interchange of two rows (or two columns) changes the sign of the determinant. Zero is the only number that is not changed when its sign is changed.

THEOREM 3. *If all the elements of a row (or column) are zero, the determinant is equal to zero.*

For every term of the expansion contains a factor from the row of zeros.

THEOREM 4. *If all the elements of a row (or column) are multiplied by a constant k, the determinant is multiplied by k.*

For every term of the expansion is multiplied by k.

COROLLARY. *If the elements of one row (or column) are proportional to the element of another row (or column), the determinant vanishes identically.*

For, when the factor of proportionality is set aside, the determinant will have two rows (or columns) alike.

THEOREM 5. *If each element of a row (or column) is made up of the sum of two numbers, the determinant can be expressed as the sum of two determinants in one of which the row of binomials is replaced by the first numbers of the sums, and in the other of which the binomials are replaced by the second members of the sums.*

Stated in symbols the theorem is given by the equation,

$$(3)\quad \begin{vmatrix} a_1+a'_1 & b_1 & c_1 & d_1 \\ a_2+a'_2 & b_2 & c_2 & d_2 \\ a_3+a'_3 & b_3 & c_3 & d_3 \\ a_4+a'_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 & d_1 \\ a'_2 & b_2 & c_2 & d_2 \\ a'_3 & b_3 & c_3 & d_3 \\ a'_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

All the terms of the first determinant are of the form

$(a_i + a'_i)b_jc_kd_l$ which is equal to $a_ib_jc_kd_l + a'_ib_jc_kd_l$,

so that each term is the sum of two, which are the corresponding terms of the last two determinants in equation (3).

COROLLARY. *A determinant is unchanged by adding to any row (or column) the corresponding elements of another row (or column) each multiplied by a common factor.* For, in the equation

$$(4)\quad \begin{vmatrix} a_1+kb_1 & b_1 & c_1 & d_1 \\ a_2+kb_2 & b_2 & c_2 & d_2 \\ a_3+kb_3 & b_3 & c_3 & d_3 \\ a_4+kb_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 & d_1 \\ kb_2 & b_2 & c_2 & d_2 \\ kb_3 & b_3 & c_3 & d_3 \\ kb_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

which follows directly from Theorem 5, the second determinant on the right is identically zero by Theorem 4.

EXERCISES

1. Prove that

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 6 \\ 8 & 4 & 2 & 9 \\ 4 & 6 & 8 & 10 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 6 & 9 & 12 & 15 \\ 3 & 2 & 1 & 6 \\ 8 & 4 & 2 & 9 \\ 4 & 6 & 8 & 10 \end{vmatrix}$$

are both zero.

2. Prove that

$$\begin{vmatrix} 2 & 3 & 4 \\ 3 & 5 & 8 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 18 & 20 & 24 \\ 24 & 12 & 3 \end{vmatrix}.$$

[HINT. First multiply each column by the quotient of the L. C. M. of the elements in the first row by the first element in the column.]

3. Express the determinant

$$\begin{vmatrix} 2 & 3 & 4 \\ 3 & 5 & 8 \\ 4 & 3 & 1 \end{vmatrix}$$

in terms of another in which the elements of the first row are 1, 0, and 0.

4. Express the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

in terms of another having a row or a column of unit elements.

5. Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \equiv (b-c)(c-a)(a-b).$$

[HINT. If b were equal to c, the determinant would vanish. Therefore $b - c$ is a factor.]

110. Minors and Cofactors. Since a determinant contains one, and only one, element from each row and each column, a determinant Δ of the fourth order may be written in the form

$$(5)\qquad \Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4,$$

where a_1A_1 stands for the algebraic sum of all the terms containing a_1 as a factor, a_2A_2 for the terms containing a_2 as a

factor, and so on. In the same way the determinant may also be written in the forms

$$(6)\qquad \begin{aligned}\Delta &= b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4\\ &= c_1C_1 + c_2C_2 + c_3C_3 + c_4C_4\\ &= d_1D_1 + d_2D_2 + d_3D_3 + d_4D_4.\end{aligned}$$

Similar statements are true for a determinant of any order.

The numbers $A_1, A_2, A_3, A_4, B_1, B_2 \cdots$ are called ***cofactors*** of the corresponding elements $a_1, a_2, a_3, a_4, b_1, b_2 \cdots$.

A closer examination of the cofactors yields valuable information. The cofactor A_1 of a_1 is made up of products of the letters b, c, d, with the subscripts taken in every possible order, and with the sign determined by the order of the subscripts 2, 3, 4, since the subscript 1 is always first. But this combination of terms is exactly the determinant

$$\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

which is obtained by striking out the first row and the first column of Δ.

To find the meaning of A_2, the first and second rows of Δ may be interchanged in order to bring a_2 to the left-hand upper corner of the determinant. The terms which are multiplied by a_2 are then seen to be exactly the terms in the expansion of the determinant,

$$\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

But considered as belonging to the original determinant Δ every sign is changed since two rows have been interchanged. Consequently A_2 is the negative of the determinant just written

In a similar way it is shown that

$$(7)\quad A_3 = +\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix},\ A_4 = -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix},\ B_1 = -\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} \cdots$$

A determinant obtained by striking out a row and a column of a determinant is called a ***complementary minor***, or simply a ***minor***, of the element that stands at the intersection of the row and the column stricken out. A minor is denoted by the letter Δ_u with the element to which it is complementary written as a subscript. Thus, Δ_{a_1}, Δ_{a_2}, $\cdots$ are the minors of a_1, a_2, $\cdots$.

From the foregoing it is clear that the cofactor is equal numerically to the minor, but that as we proceed from the left-hand upper corner, along either rows or columns, the alternate minors, beginning with the second, differ in sign from the corresponding cofactors. The rule is that *the minor corresponding to the element in the ith row and the kth column is equal to the cofactor, or to the cofactor with its sign changed, according as $i + k$ is even or odd.*

The reason for the rule is easily seen to be the fact that the sign of the original determinant is unchanged or not, by bringing the element in the ith row and kth column to the upper left-hand corner, according as the number of interchanges $i + k$ is even or odd.

Using the minors instead of the cofactors we have for the determinant (1) of fourth order

$$(8)\qquad \begin{aligned} \Delta &\equiv a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4} \\ &\equiv - b_1\Delta_{b_1} + b_2\Delta_{b_2} - b_3\Delta_{b_3} + b_4\Delta_{b_4} \\ &\equiv c_1\Delta_{c_1} - c_2\Delta_{c_2} + c_3\Delta_{c_3} - c_4\Delta_{c_4} \\ &\equiv - d_1\Delta_{d_1} + d_2\Delta_{d_2} - d_3\Delta_{d_3} + d_4\Delta_{d_4}. \end{aligned}$$

In this way we have $2n$ different expressions for a determinant in terms of its minors, or of its cofactors.

Almost as important as the foregoing expressions for a determinant, are the identities that exist between its elements.

THEOREM 6. *The sum of the products of the elements of a row (or column) each multiplied by the cofactor of the corresponding element of another row (or column) is identically zero.*

For example, since

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4,$$

the expression

$$b_1A_1 + b_2A_2 + b_3A_3 + b_4A_4,$$

is a determinant obtained from Δ by replacing $a_1, a_2, \ldots$ by $b_1, b_2, \ldots$, and, consequently, has two columns identical. Therefore

$$(9) \qquad \boldsymbol{b_1A_1 + b_2A_2 + b_3A_3 + b_4A_4 \equiv 0.}$$

Many other identities of this type may be written out. There are altogether 24 of them for a determinant of order 4.

111. Computation of Determinants.

The computation of a determinant whose elements are known numbers is best illustrated by concrete examples.

EXAMPLE 1. Compute the determinant

$$\Delta = \begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix}.$$

If from the first, second, and third columns, twice, three times, and four times the last column be subtracted, according to Theorem 5, Cor. I, the result will be

$$\Delta=\begin{vmatrix}1&1&1&4\\2&4&1&8\\4&1&2&13\\2&4&2&11\end{vmatrix}=\begin{vmatrix}1&1&1&1\\2&4&1&1\\4&1&2&6\\2&4&2&3\end{vmatrix}.$$

The last determinant is obtained by subtracting the sum of the first three columns from the fourth. Subtracting the third column from the other three in turn, and then noting that the sum of the products of the elements in the first row by the corresponding minor contains a single term, we have

$$\Delta=\begin{vmatrix}0&0&1&0\\1&3&1&0\\2&-1&2&4\\0&2&2&1\end{vmatrix}=\begin{vmatrix}1&3&0\\2&-1&4\\0&2&1\end{vmatrix}.$$

The last determinant is reduced to a determinant of order 2 by subtracting twice the first row from the second, and then striking out the first row and the first column. We find

$$\Delta=\begin{vmatrix}1&3&0\\0&-7&4\\0&2&1\end{vmatrix}=\begin{vmatrix}-7&4\\2&1\end{vmatrix}=-15.$$

EXAMPLE 2. Compute the determinant

$$\Delta=\begin{vmatrix}5&-10&11&0\\-10&-11&12&4\\11&12&-11&2\\0&4&2&-6\end{vmatrix}.$$

Adding the sum of the second and third rows to the fourth, and subtracting twice the third from the second, we find

$$\Delta=\begin{vmatrix}5&-10&11&0\\-32&-35&34&0\\11&12&-11&2\\1&5&3&0\end{vmatrix}=-2\begin{vmatrix}5&-10&11\\-32&-35&34\\1&5&3\end{vmatrix}=-10\begin{vmatrix}5&-2&1\\-32&-7&-1\\1&1&8\end{vmatrix}.$$

Adding the first row to the second, then subtracting 8 times the first from the third, we find

$$\Delta=+10\begin{vmatrix}5&-2&1\\27&9&0\\-39&17&0\end{vmatrix}=90\begin{vmatrix}3&1\\-39&17\end{vmatrix}=8100.$$

In the foregoing, the orders of the determinants have been reduced by bringing them to such a form that all elements but one in a row or a column are zero. This process may always be accomplished by multiplying the elements of the columns, say, by such factors that the elements in a row will be replaced by a common multiple. All factors introduced in this way must be taken out again by division. For example,

$$\begin{vmatrix} 2 & 3 & 1 \\ 5 & 7 & 8 \\ 6 & 3 & 4 \end{vmatrix} = \frac{1}{3 \cdot 2 \cdot 6}\begin{vmatrix} 6 & 6 & 6 \\ 15 & 14 & 48 \\ 18 & 6 & 24 \end{vmatrix} = \frac{1}{3 \cdot 2}\begin{vmatrix} 1 & 1 & 1 \\ 15 & 14 & 48 \\ 18 & 6 & 24 \end{vmatrix}$$

$$= \frac{1}{3 \cdot 2}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 14 & 34 \\ 12 & 6 & 18 \end{vmatrix} = -\frac{1}{3 \cdot 2}\begin{vmatrix} 1 & 34 \\ 12 & 18 \end{vmatrix} = \begin{vmatrix} 1 & 34 \\ 2 & 3 \end{vmatrix}.$$

EXERCISES

1. Compute the values of the following determinants.

(a) $\begin{vmatrix} 6 & 12 & 25 \\ 8 & 20 & 25 \\ 10 & 20 & 30 \end{vmatrix}$.

(b) $\begin{vmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 3 \\ 5 & 6 & 3 & 4 \\ 6 & 3 & 4 & 5 \end{vmatrix}$.

(c) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$.

(d) $\begin{vmatrix} 3 & 5 & 7 & 9 \\ 3 & 6 & 9 & 7 \\ 4 & 1 & 7 & 8 \\ 3 & 8 & 7 & 4 \end{vmatrix}$.

(e) $\begin{vmatrix} 5 & 3 & 4 & 2 \\ 5 & 4 & 2 & 3 \\ 0 & 3 & 6 & 7 \\ 0 & 0 & 6 & 3 \end{vmatrix}$.

(f) $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 5 & 7 & 8 \\ 3 & 4 & 6 & 3 \end{vmatrix}$.

(g) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

(h) $\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix}$.

2. Express the determinant

$$\begin{vmatrix} a^3 & b^3 & c^3 \\ (a+\lambda)^3 & (b+\lambda)^3 & (c+\lambda)^3 \\ (2a+\lambda)^3 & (2b+\lambda)^3 & (2c+\lambda)^3 \end{vmatrix}$$

as a polynomial in λ. [SALMON, *Higher Algebra*.]

3. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{a_1 b_1 c_1} \begin{vmatrix} 1 & 1 & 1 \\ a_2 b_1 c_1 & b_2 c_1 a_1 & c_2 a_1 b_1 \\ a_3 b_1 c_1 & b_3 c_1 a_1 & c_3 a_1 b_1 \end{vmatrix}.$$

Show how the result can be utilized in the computation of determinants whose elements are known numbers.

4. Prove by the factor theorem that $x - \alpha$ and $x - \beta$ are factors of

$$\begin{vmatrix} 1 & 1 & 1 \\ x & \alpha & \beta \\ x^2 & \alpha^2 & \beta^2 \end{vmatrix}.$$

5. Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)[L\alpha + M\beta + N\gamma].$$

[HINT. Use the method of Problem 4, then note degree of determinant and the coefficients of three terms suitably chosen.]

6. Find the expression for the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}.$$

7. Find the expression for the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^4 & \beta^4 & \gamma^4 \end{vmatrix}.$$

[HINT. The result must be of degree five and can change sign only when any two of the numbers α, β, γ, are interchanged.]

8. The area of a triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3), is given by the formula

$$\text{Area} = \tfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Find the area of the triangle whose vertices are at the points (3, 2), (5, 6), (4, − 7).

9. The volume of a tetrahedron whose vertices are at the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4), is given by

$$\text{Volume} = \frac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

Find the volume of the tetrahedron whose vertices are at the points (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1).

112. Application to Linear Equations.

Let

$$(10) \qquad \begin{cases} a_1x + b_1y + c_1z + d_1w = e_1, \\ a_2x + b_2y + c_2z + d_2w = e_2, \\ a_3x + b_3y + c_3z + d_3w = e_3, \\ a_4x + b_4y + c_4z + d_4w = e_4, \end{cases}$$

be a system of four linear equations in four unknowns. The determinant,

$$\Delta = (a_1b_2c_3d_4),$$

is called the ***determinant of the system.*** If the first equation be multiplied by A_1, the second by A_2, the third by A_3, and the fourth by A_4, where the A's are cofactors, and the resulting equations added, the coefficient of x will be exactly equal to Δ, by (6), § 110. On the other hand, the coefficients of y, z, and w will be zero, by (9), § 110. We have, therefore,

$$\Delta x = e_1A_1 + e_2A_2 + e_3A_3 + e_4A_4.$$

The expression on the right is Δ with e's substituted for a's. Consequently

$$(11) \qquad x = \frac{\begin{vmatrix} e_1 & b_1 & c_1 & d_1 \\ e_2 & b_2 & c_2 & d_2 \\ e_3 & b_3 & c_3 & d_3 \\ e_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\Delta}.$$

Similarly,

$$(12)\quad y=\frac{\begin{vmatrix} a_1 & e_1 & c_1 & d_1 \\ a_2 & e_2 & c_2 & d_2 \\ a_3 & e_3 & c_3 & d_3 \\ a_4 & e_4 & c_4 & d_4 \end{vmatrix}}{\Delta},\quad z=\frac{\begin{vmatrix} a_1 & b_1 & e_1 & d_1 \\ a_2 & b_2 & e_2 & d_2 \\ a_3 & b_3 & e_3 & d_3 \\ a_4 & b_4 & e_4 & d_4 \end{vmatrix}}{\Delta},\quad w=\frac{\begin{vmatrix} a_1 & b_1 & c_1 & e_1 \\ a_2 & b_2 & c_2 & e_2 \\ a_3 & b_3 & c_3 & e_3 \\ a_4 & b_4 & c_4 & e_4 \end{vmatrix}}{\Delta}.$$

This solution of the system is simply an extension of Cramer's rule (§§ 70 and 73) to four equations. Clearly, the method applies to any number of linear equations, provided the number of unknowns is equal to the number of equations, and *the determinant of the system is not zero.*

EXERCISES

1. Solve the following systems of equations.

$$(a)\quad \begin{cases} x+y+z=6, \\ 3x-y+2z=7, \\ 4x+3y-z=7. \end{cases}$$

$$(b)\quad \begin{cases} 3x+4y-5z=32, \\ 4x-5y+3z=18, \\ 5x-3y-4z=2. \end{cases}$$

$$(c)\quad \begin{cases} x-9y+3z-10w=-102, \\ 2x+7y-z-w=51, \\ 3x+y+5z+2w=70, \\ 4x-6y-2z-9w=-67. \end{cases}$$

$$(d)\quad \begin{cases} -x+y+z+w=a, \\ x-y+z+w=b, \\ x+y-z+w=c, \\ x+y+z-w=d. \end{cases}$$

$$(e)\quad \begin{cases} x+y-2z+w=10, \\ 2x-y+z-w=-7, \\ x+3y+2z-w=0, \\ 3x+y+3z+2w=19. \end{cases}$$

$$(f)\quad \begin{cases} x+y=4, \\ y+z=6, \\ z+x=8. \end{cases}$$

$$(g)\quad \begin{cases} x+y=1, \\ y+z=2, \\ z+w=3, \\ w+x=2. \end{cases}$$

$$(h)\quad \begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{y}+\dfrac{1}{z}=b, \\ \dfrac{1}{z}+\dfrac{1}{x}=c. \end{cases}$$

2. Find the coefficients A, B, C, and D, which will make the curve

$$y = Ax^3 + Bx^2 + Cx + D$$

pass through the points (0, 1), (1, 1), (2, 0), (3, 1).

113. Homogeneous Linear Equations. A system of linear equations in which every absolute term is zero is called a ***homogeneous system.*** A system of four homogeneous linear equations may be written as follows:

$$(13)\qquad \begin{cases} a_1x + b_1y + c_1z + d_1w = 0, \\ a_2x + b_2y + c_2z + d_2w = 0, \\ a_3x + b_3y + c_3z + d_3w = 0, \\ a_4x + b_4y + c_4z + d_4w = 0. \end{cases}$$

Let $\Delta = (a_1b_2c_3d_4)$ be the determinant of the system; then by Cramer's rule,

$$(14)\qquad \Delta x = 0,\ \Delta y = 0,\ \Delta z = 0,\ \Delta w = 0.$$

Therefore, if $\Delta \neq 0$, we must have

$$(15)\qquad x = 0,\ y = 0,\ z = 0,\ w = 0.$$

This solution, which might have been obtained by inspection of the system (13), is of little importance and is called the ***trivial solution.***

It is also clear from (14) that no solutions other than the trivial solution can exist *unless* $\Delta = 0$. This demonstration applies to n equations in n unknowns.

Therefore the necessary condition that n homogeneous linear equations in n unknowns have a solution other than the trivial solution, is that the determinant of the system shall be zero.

The determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is called the ***eliminant*** of the homogeneous system (13).

114. The use of Eliminants in Geometry. Suppose, for example, that it is required to find the equation of a straight line which passes through two given points $(x_1, y_1,)$ and (x_2, y_2).

According to § 49 the equation of the line has the form

$$(16)\qquad Ax + By + C = 0$$

where A, B, and C are undetermined coefficients. Since the coördinates of the points (x_1, y_1), and (x_2, y_2), satisfy equation (16) we have the system

$$(17)\qquad \begin{cases} Ax + By + C = 0, \\ Ax_1 + By_1 + C = 0, \\ Ax_2 + By_2 + C = 0, \end{cases}$$

from which to determine A, B, and C. The values of A and B in the terms of C might be determined from the last two equations of the system (17). These values could be inserted in the first equation, and the required equation would then be obtained by dividing by C.

The same result may be obtained directly by looking upon the system (17) as a system of equations homogeneous in A, B, and C. According to § 113 if A, B, and C are not all zero, we must have

$$(18)\qquad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Equation (18) is the equation of a straight line since it is a linear equation in two variables. Moreover, it is satisfied by the coördinates of both the points (x_1, y_1) and (x_2, y_2), since, if the coördinates of either point be substituted for x and y, the determinant will vanish identically.

Example. The equation of the line through the two points (2, 3) and (3, 5), is

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{vmatrix} = 0,$$

or

$$2x - y - 1 = 0.$$

EXERCISES

1. Assuming the equation of a plane to be of the form

$$Ax + By + Cz + D = 0,$$

find the equation in determinant form of the plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

2. What is the equation of the line through the points (2, 3) and (5, 7)?

3. What is the equation of the plane through the three points (3, 5, 1), (4, 5, 6), (4, −5, 6)?

4. The equation of a circle has the form (§ 76)

$$x^2 + y^2 + Ax + By + C = 0.$$

What is the equation of the circle through the three points (1, 0), (0, 1), (1, 1)?

115. Consistency of $n + 1$ Non-homogeneous Linear Equations. In § 112 it was shown that for a system of n non-homogeneous equations whose determinant is not zero, values of the n unknowns may be found, which will satisfy the equations. If another equation be added to the system, the solution found for the first n equations will not, in general, satisfy it. It is possible, however, to think of the new system as a system of homogeneous linear equations in $n + 1$ unknowns simply by writing 1 for the $(n + 1)$st unknown. Thus the system of three equations,

$$(19)\qquad \begin{cases} a_1x + b_1y = c_1 \cdot 1, \\ a_2x + b_2y = c_2 \cdot 1, \\ a_3x + b_3y = c_3 \cdot 1, \end{cases}$$

is homogeneous in x, y, and 1. The necessary condition that this system should be consistent is

$$(20)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Geometrically considered, (20) is the condition that the three lines given by (19) meet in a point.

EXERCISES

1. Show that the three lines

$$\begin{cases} 2x + 5y = 7, \\ 3x - 4y = -1, \\ 5x + 3y = 8, \end{cases}$$

meet in a point. What is the point?

2. Show that the three lines

$$\begin{cases} 3x + 2y = 8, \\ 4x - y = 7, \\ 5x + 3y = 8, \end{cases}$$

do not pass through a common point.

3. Show that the four planes

$$\begin{cases} 2x + 3y - z = 1, \\ 3x - 4y + z = -1, \\ 6x - 4y + 3z = -3, \\ 3x + 5y - 5z = 5, \end{cases}$$

meet in a point.

116. Homogeneous Systems of $n - 1$ Equations. Let

$$(21) \qquad \begin{cases} a_1x + b_1y + c_1z + d_1w = 0, \\ a_2x + b_2y + c_2z + d_2w = 0, \\ a_3x + b_3y + c_3z + d_3w = 0, \end{cases}$$

be a system of three homogeneous linear equations in four unknowns. The rectangular array

$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{Vmatrix}$$

written with *two* bars at each side, is called the ***matrix*** of the system (21).

THEOREM. *The values of the n unknowns which satisfy $n-1$ homogeneous linear equations are proportional to the n determinants with signs alternately minus and plus, obtained by striking out the first, the second, ⋯ and the nth columns of the matrix of the system.**

The method of proof for three equations holds for any number.

Let

$$\Delta = (a_1 b_2 c_3 d_4)$$

be the determinant of the system (13), § 113, and let A_4, B_4, C_4, D_4, be the cofactors corresponding to the elements a_4, b_4, c_4, d_4 in Δ. Then, by § 110,

$$A_4 = -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix},\ B_4 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix},\ C_4 = -\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

$$D_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

where it is to be observed that A_4, B_4, C_4, D_4, are formed according to the statement of the theorem. Solving the system (21) for x, y, z, in terms of w, by Cramer's rule, we find

$$(22)\qquad D_4 x = -\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} w,\quad D_4 y = -\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} w,$$

$$D_4 z = -\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} w.$$

* It is assumed in the proof that the determinant $(a_1 b_2 c_3) \neq 0$, though this is not necessary. (See BÔCHER, *Higher Algebra*, The Macmillan Company, p. 47.)

But the determinants on the right become, after suitable interchanges of columns, exactly A_4, B_4, C_4, so that

$$(23)\qquad D_4x = A_4w, \quad D_4y = B_4w, \quad D_4z = C_4w,$$

or

$$(23a)\qquad \frac{x}{A_4} = \frac{y}{B_4} = \frac{z}{C_4} = \frac{w}{D_4}.$$

117. Two Equations having a Common Root. Let the two equations with one unknown be

$$(24)\qquad \begin{cases} a_0x^2 + a_1x + a_2 = 0, \\ b_0x^3 + b_1x^2 + b_2x + b_3 = 0. \end{cases}$$

Multiply the first by x^2 and by x, and the second by x. In this way five equations are obtained,

$$\begin{aligned} a_0x^4 + a_1x^3 + a_2x^2 \qquad\qquad &= 0, \\ a_0x^3 + a_1x^2 + a_2x \qquad &= 0, \\ a_0x^2 + a_1x + a_2 &= 0, \\ b_0x^4 + b_1x^3 + b_2x^2 + b_3x \qquad &= 0, \\ b_0x^3 + b_1x^2 + b_2x + b_3 &= 0. \end{aligned}$$

If these five equations are looked upon as five homogeneous equations with the unknowns x^4, x^3, x^2, x, 1, the necessary condition that they have a common solution is, by § 113,

$$(25)\qquad \begin{vmatrix} a_0 & a_1 & a_2 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

The determinant in equation (25) is called the ***resultant***, or the ***eliminant***, of the equations (24). The resultant of two equations of degrees m and n is a determinant of order $m + n$.

Proofs of the sufficiency of the condition may be found in FINE, *College Algebra*, and in DICKSON, *Elementary Theory of Equations*.

EXERCISES

1. Show that the condition that the equations

$$a_0x + a_1 = 0, \text{ and } b_0x + b_1 = 0,$$

have a common root is $a_0b_1 - a_1b_0 = 0$.

2. Find the condition that two quadratic equations

$$a_0x^2 + a_1x + a_2 = 0 \text{ and } b_0x^2 + b_1x + b_2 = 0$$

shall have a common root.

CHAPTER XIII

THE BINOMIAL THEOREM

118. Statement of the Theorem. The binomial theorem is a theorem which enables us to write out at once any power of a binomial. For a positive integral exponent n the theorem may be expressed by the following formula:

$$(1)\quad (x+a)^n = x^n + \frac{n}{1}x^{n-1}a + \frac{n(n-1)}{1\cdot 2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}a^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}x^{n-r+1}a^{r-1} + \cdots + a^n.$$

Proof. For $n = 2$,

$$(x+a)^2 = x^2 + \frac{2}{1}xa + \frac{2(2-1)}{1\cdot 2}a^2 = x^2 + 2xa + a^2.$$

Similarly, for $n = 3$,

$$(x+a)^3 = x^3 + 3x^2a + \frac{3(3-1)}{1\cdot 2}xa^2 + \frac{3(3-1)(3-2)}{1\cdot 2\cdot 3}a^3 = x^3 + 3x^2a + 3xa^2 + a^3,$$

which again agrees with the result of direct multiplication. In this way we are led to suspect the truth of the theorem as stated. Verification is obviously impossible for more than a few values of n. The proof for any positive integral exponent is obtained by a method called ***mathematical induction.*** By this method it is shown that if equation (1) is true for any particular value of n, it is true for the next larger value; and, consequently, since it is known to be true for one or two small values, it is true for every positive integral value of n.

Suppose the theorem is true for $n = k$. The hypothesis is then

(2) $$(x+a)^k = x^k + kx^{k-1}a + \frac{k(k-1)}{1 \cdot 2} x^{k-2}a^2 + \cdots + \frac{k(k-1) \cdots (k-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} x^{k-r+1}a^{r-1} + \cdots + a^k.$$

If both sides be multiplied by $x + a$, the result is

$$(x+a)^{k+1} = x^{k+1} + kx^k a + \frac{k(k-1)}{1 \cdot 2} x^{k-1}a^2 + \cdots$$
$$+ \frac{k(k-1) \cdots (k-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} x^{k-r+2}a^{r-1} + \cdots + xa^k$$
$$+ x^k a + kx^{k-1}a^2 + \cdots + \frac{k(k-1) \cdots (k-r+3)}{1 \cdot 2 \cdot 3 \cdots (r-2)} x^{k-r+2}a^{r-1}$$
$$+ \cdots + a^{k+1}.$$

When like terms on the right are combined, it will be seen that

the coefficient of $x^k a$ is $k+1$;

the coefficient of $x^{k-1}a^2$ is $\frac{k(k-1)}{1 \cdot 2} + k = \frac{(k+1)k}{1 \cdot 2}$;

the coefficient of $x^{k-r+2}a^{r-1}$ is

$$\frac{k(k-1) \cdots (k-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} + \frac{k(k-1) \cdots (k-r+3)}{1 \cdot 2 \cdot 3 \cdots (r-2)},$$

which reduces to $\frac{(k+1)k(k-1) \cdots (k-r+3)}{1 \cdot 2 \cdot 3 \cdots (r-1)}$.

Moreover, $k, k-1, k-2, \cdots$ may be written in the forms $k+1-1, k+1-2, k+1-3, \cdots$. Consequently,

$$(x+a)^{k+1} = x^{k+1} + (k+1)x^{k+1-1}a + \frac{(k+1)(k+1-1)}{1 \cdot 2} x^{k+1-2}a^2$$
$$+ \cdots + \frac{(k+1)(k+1-1)(k+1-2) \cdots (k+1-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} x^{k+1-r+1}a^{r-1}$$
$$+ \cdots + a^{k+1}.$$

The last formula is exactly like (2) except that everywhere k in (2) is replaced by $k+1$. In other words, if the theorem holds for $n = k$, it holds also for $n = k+1$. This proves that

if the theorem is true for one value of n, it is true for the next larger value of n. But it is true for $n=3$, then for $n=4$, then for $n=5$, and so on. It is therefore true for any positive integral value of n, as was to be proven.

The expansion of $(x-a)^n$ is found directly by writing it in the form $[x+(-a)]^n$. It follows that all the terms containing odd powers of $-a$, that is, the first, third, fifth, powers and so on, are negative, while all other terms are positive.

119. The Number of Terms and the *r*th Term. When n is a positive integer, the number of terms in the expansion o, $(x+a)^n$ is $n+1$, since the first term, which does not contain a, is followed by the terms containing the first n consecutive powers of a.

The $n+1$ numbers

$$(3)\qquad 1,\ \frac{n}{1},\ \frac{n(n-1)}{1\cdot 2},\ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3},\ \cdots,\ \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)},\ \cdots,\ 1,$$

are called the ***binomial coefficients***, and are frequently denoted by the symbols

$$(4)\qquad {}_nC_0,\ {}_nC_1,\ {}_nC_2,\ \cdots,\ {}_nC_r,\ \cdots,\ {}_nC_n.$$

The rth coefficient is ${}_nC_{r-1}$, and the rth term is

$$(5)\qquad {}_nC_{r-1}x^{n-r+1}a^{r-1} = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}x^{n-r+1}a^{r-1}.$$

The product of the n *consecutive integers* beginning with 1 is called ***factorial n***, and is written $n!$, or $|\underline{n}$.

By definition the denominator of the rth term is $(r-1)!$ If both numerator and denominator be multiplied by the product of the factors

$$(n-r+1),\ n-r,\ n-r-1,\ \cdots,\ 3,\ 2,\ 1,$$

that is, by $(n-r+1)!$, the numerator becomes the product of the n integers from n down to 1 inclusive. It will then be $n!$ The coefficient of the rth term may then be written in the form,

$$_nC_{r-1} = \frac{n!}{(r-1)!(n-r+1)!},$$

and the expansion itself takes the form,

$$(x+a)^n = x^n + \frac{n!}{1!(n-1)} x^{n-1}a + \frac{n!}{2!(n-2)!} x^{n-2}a^2 + \cdots$$
$$+ \frac{n!}{(r-1)!(n-r+1)!} x^{n-r+1}a^{r-1} + \cdots + a^n.$$

The most important property of factorial n is expressed by the identity

$$n! = n \times (n-1)!$$

120. Pascal's Triangle. The following triangular array of numbers formed by adding two adjacent numbers in a line and writing the sum under the one on the right is called ***Pascal's Triangle:***

```
1
1  1
1  2   1
1  3   3   1
1  4   6   4   1
1  5  10  10   5  1
1  6  15  20  15  6  1
.  .   .   .   .  .  .  .  .  .
```

The numbers in the first few horizontal lines of the triangle are seen at once to be the binomial coefficients of $(x+a)^0$, $(x+a)^1$, $(x+a)^2$, and so on. That the numbers in any line are the coefficients of a power of $x+a$ is easily shown by adding two consecutive numbers in a line, say $_nC_{r-1}$ and $_nC_r$, and showing that the sum reduces to $_{n+1}C_r$. The proof is left to the student.

EXERCISES

1. Expand the following binomials.

(a) $(x+y)^6$.

(b) $(x-y)^6$.

(c) $(x-2y)^5$.

(d) $(a-3b)^5$.

(e) $\left(x+\frac{1}{x}\right)^5$.

(f) $(x-2\sqrt{y})^6$.

(g) $(a^{\frac{2}{3}}+3bc^{\frac{1}{2}})^6$.

(h) $\left(2-\frac{3}{\sqrt{9}}\right)^4$.

(i) $(\sqrt[3]{a}-\sqrt[6]{b})^6$.

(j) $(1.03)^2$.

(k) $(1.025)^2$.

(l) $(1.01)^4$.

(m) $(53)^2$.

(n) $(69)^2$.

(o) $(84)^2$.

(p) $(x+[y+z])^4$.

(q) $(x^2-2ax+a^2)^4$.

(r) $(x^2-2ax+2a^2)^8$.

2. Write down the 11th term of $(a+x)^{15}$.

3. Write down the 12th term of $(z-x)^{17}$.

4. Write the middle term of $(x-2y)^{18}$.

5. Prove that the coefficient of $a^rb^sc^t$ in the expansion of

$$(a+b+c)^n$$

is $n!/(r!s!t!)$.

[HINT. The $(t+1)$st term of $([a+b]+c)^n$ is

$$\frac{n!}{t!(n-t)!}(a+b)^{n-t}c^t.]$$

6. From the result of Ex. 5, find the coefficient of x^6 in the expansion $(1+2x+x^2)^{35}$.

7. Prove that ${}_nC_r={}_nC_{n-r}$.

8. Taking the equation $n!=n(n-1)!$ as a defining equation, prove that $0!=1$. (Compare with $a^0=1$.)

9. Find the ratio of the $(n+1)$st to the nth term, and from the result derive a rule by means of which any term of the binomial expansion may be found from the preceding one.

10. Prove that the algebraic sums of the binomial coefficients in the expansions of $(x+a)^n$ and $(x-a)^n$ are 2^n and 0 respectively.

[HINT. Formula (1) of § 118 is an identity and is therefore true for any values of x and a.]

11. Find the sums of the numerical, as distinguished from the binomial, coefficients in the expansions for $(x+2y)^7$ and $(x-2y)^7$.

121. The Binomial Theorem when n is not a Positive Integer. In a later section it is proven that the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots,$$

where the continuation marks indicate that the process is to be carried on indefinitely, holds true for all values of x which are less in absolute value than 1. For present purposes the chief value of such an expansion consists in the fact that a few terms will sometimes give, with little labor, *approximate results* which might otherwise be difficult to obtain.

EXAMPLE. Find the square root of 1.01.

By the above expansion we have

$$\begin{aligned}(1.01)^{1/2} &= 1 + \tfrac{1}{2}.01 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}(.01)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}(.01)^3 + \cdots \\ &= 1 + .005 - .0000125 + .0000000625 + \cdots \\ &= 1.0049875625\cdots.\end{aligned}$$

This result is correct to the 8th decimal place.

EXERCISES

1. Compute the numerical values of each of the following quantities.

   (a) $(1.02)^{1/2}$. (b) $(1.02)^{1/4}$. (c) $\dfrac{1}{(1.01)^2}$.

2. Extract the 6th root of 70 by finding the value of

$$(64+6)^{1/6} = 64^{1/6}(1+\tfrac{3}{32})^{1/6}.$$

3. The formula $A' = A(1+i)^n$ for compound interest is assumed to hold for n less than one year. Find the compound interest on \$1000 for one month at 4 per cent. Compare your result with the simple interest on the same principal for the same time at the same rate.

4. The expression $p[(1+i)^{1/p}-1]$ where i is rate of interest and p the number of times interest is compounded in a year, is important in the theory of investment. Find its value when $i = .04$ and $p = 4$.

5. When the rate of interest is i, the present value of a sum due n years hence without interest is $A(1+i)^{-n}$. Find the present value of \$1000 due in 5 yr. without interest if money is worth 5 %.

CHAPTER XIV

PROGRESSIONS. — COMPOUND INTEREST

122. Arithmetic Progressions. A set of numbers like

$$3,\ 7,\ 11,\ 15,\ 19,$$

such that any one after the first may be obtained by adding a constant, is called an ***arithmetic progression.*** The constant is called the ***common difference*** of the progression.

If the common difference is positive, as in the example just given, the progression is said to be *increasing;* if it is negative, as in

$$21,\ 18,\ 15,\ 12,\ 9,$$

the progression is said to be *decreasing.*

There are five fundamental numbers belonging to every progression. These numbers are *the first term, the common difference, the number of terms, the last term,* and *the sum of all the terms.* They are denoted by

$$a_1,\ d,\ n,\ a_n, \text{ and } s_n,$$

respectively. The terms between a_1 and a_n are called the ***means.*** Any arithmetic progression

$$a_1,\ a_2,\ a_3,\ \cdots,\ a_n,$$

takes the form

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ \cdots.$$

The *general* or *type* term has the form $a_1 + d \cdot x$ where x is a positive integer. Consequently, the values of any linear integral function (§ 48) which correspond to integral values of the variable, taken at equal intervals, form an arithmetic progression.

123. The *n*th Term. To find the nth term of an arithmetic progression, we note that, so far as we can see, the coefficient of d is one less than the number of the term. We may assert, therefore, that the nth term is given by the formula

$$a_n = a_1 + (n-1)d. \tag{1}$$

The proof is easily completed by mathematical induction. For, suppose that (1) is true for $n = k$. Then by hypothesis,

$$a_k = a_1 + (k-1)d. \tag{2}$$

By the law of formation for the progression,

$$a_{k+1} = a_k + d = a_1 + (k-1)d + d, \text{ or, } a_{k+1} = a_1 + (k+1-1)d.$$

But the last equation kiffers from (2) only in the substitution of $k+1$ for k. It follows, therefore, that if the formula (1) is true for $n = k$ it is true for $n = k+1$ and consequently for any value of n.

If formula (1) is solved for a_1, the result is

$$a_1 = a_n - (n-1)d, \tag{3}$$

which expresses the first term in terms of the last term, the common difference and the number of terms.

124. The Sum of *n* Terms. To find the sum s_n of n terms, note that the sum is not changed if the terms be written in reverse order. Consequently, the sum s_n may be written in the two forms,

$$s_n = a_1 + a_1 + d + a_1 + 2d + a_1 + 3d + \cdots + a_1 + (n-1)d,$$
$$s_n = a_n + a_n - d + a_n - 2d + a_n - 3d + \cdots + a_n - (n-1)d.$$

When these two equations are added, we find

$$2s_n = n(a_1 + a_n).$$

Consequently,

$$s_n = \frac{n}{2}(a_1 + a_n). \tag{4}$$

125. The Solution of Problems. The two formulas, (1) and (4), involve the five fundamental numbers. If, therefore,

any three are given, the other two may by found by considering these two formulas as a system of two simultaneous equations with the two required numbers as unknowns.

EXERCISES

1. Given $d = 2$, $n = 12$, $s_n = 168$. Find a_1 and a_n.

SOLUTION. Substituting in formulas (1) and (4) the known values $d = 2$, $n = 12$, and $s_n = 168$, we have

$$a_n = a_1 + (12 - 1)2, \quad 168 = \tfrac{12}{2}(a_1 + a_n).$$

From these two simultaneous equations we find $a_1 = 3$ and $a_n = 25$.

2. Find the unknown numbers in each of the following cases.

(*a*) $a_1 = 3,\ d = 4,\ n = 21.$

(*b*) $a_1 = 2,\ a_n = 1460,\ n = 7.$

(*c*) $d = 3,\ a_n = 30,\ s_n = 81.$

(*d*) $d = 4,\ a_n = 43,\ s_n = 133.$

(*e*) $a_1 = 27,\ d = 7,\ n = 30.$

(*f*) $a_1 = \frac{2}{3},\ d = \frac{1}{12},\ n = 8.$

3. Find the nth term of the progression 1, 3, 5, 7, ⋯ which is composed of the first n odd positive integers.

4. Prove that the sum of n terms of the progression 1, 3, 5, 7, ⋯ is a perfect square for all values of n.

5. Find the nth even number of the series of which 2 is the first term.

6. Find the sum of the first n positive integral multiples of 3.

7. Find the sum of n terms of the progression $2 + 2\frac{1}{3} + 2\frac{2}{3} + \cdots$.

8. Find the sum of n terms of the progression

$$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \cdots.$$

9. If a body falls 16.1 feet the first second, 48.3 the second, 80.5 feet the third, how far will it fall in half a minute?

10. Fifty potatoes are arranged in a line three feet apart and the first is three feet from a basket. How far will a contestant travel who picks up and carries each potato singly to the basket?

11. How many terms of the progression .034, .0344, .0348, ⋯ will be required for the sum to amount to 2.748?

12. Insert 9 arithmetic means between 100 and 200.

13. What is the sum of 15 terms of the arithmetic progression whose first two terms are $x^2 - 14x + 1$ and $x^2 - 12x + 1$?

14. The 6th term of an arithmetic progression is 32 and 11th is 57. What is the sum of the first 9 terms?

15. Prove that if the terms of an arithmetic progression are multiplied by any constant the products form an arithmetic progression.

16. The terms of an arithmetic progression containing rn terms are grouped together in r sets of n consecutive terms and the sum of each group is found. Prove that the sums form an arithmetic progression. If the common difference of the original series is d, what is the common difference of the new series?

17. Prove that the sum of any $2n + 1$ consecutive integers is divisible by $2n + 1$.

18. Prove that if the numbers of the terms be taken as abscissas and the terms themselves as ordinates, the points determined by an arithmetic progression will lie in a straight line. What is the geometric significance of the common difference?

126. Geometric Progressions. A ***geometric progression*** is a set of numbers $a_1, a_2, \cdots, a_n$, such that any one after the first may be obtained from the preceding number by multiplying by a constant factor. The constant factor is called the ***common ratio*** or, simply the ***ratio***.

A geometric progression like

$$1,\ 2,\ 4,\ 8,$$

or like

$$1,\ -2,\ 4,\ -8,$$

where the absolute value of the ratio is greater than 1, is said to be *increasing*. If the absolute value of the ratio is less than one, as in

$$3,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{9},$$

and in

$$4,\ -2,\ 1,\ -\tfrac{1}{2},$$

the progression is said to be *decreasing*.

As in arithmetic progressions, there are five fundamental numbers to be considered, namely, the first term a_1, the ratio r,

the number of terms n, the nth term a_n, and the sum of n terms s_n. With this notation the progression

$$a_1, \quad a_2, \quad a_3, \quad a_4, \cdots,$$

may be written in the form

$$a_1, \quad a_1r, \quad a_1r^2, \quad a_1r^3, \cdots,$$

where the general, or type term, has the form a_1r^x.

127. The *n*th Term. By inspection, we observe that the nth term of the progression is given by the formula

$$a_n = a_1r^{n-1}. \tag{5}$$

That this formula is correct, may be proved rigorously by a mathematical induction similar to that employed in finding the formula for the nth term of an arithmetic progression.

128. The Sum of *n* Terms. If we multiply both sides of the equation

$$s_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}$$

by the ratio r, we obtain a second geometric progression whose sum is

$$rs_n = a_1r + a_1r^2 + \cdots + a_1r^{n-1} + a_1r^n.$$

When the first of these equations is subtracted from the second, all terms on the right disappear except a and ar^n. The resulting equation is

$$(r-1)s_n = a_1r^n - a_1.$$

Consequently,

$$s_n = \frac{a_1r^n - a_1}{r-1}. \tag{6}$$

This formula may be put in a somewhat more convenient form by replacing ar^n by its value ra_n, which is readily obtained by multiplying both sides of (5) by r. The value of s_n is then given by the formula

$$s_n = \frac{ra_n - a_1}{r-1}. \tag{6 a}$$

129. The Solution of Problems. All problems arising in geometric progressions may be solved by means of formulas (5) and (6), or (5) and (6 a), looked upon as a pair of simultaneous equations in two unknowns. Problems requiring the number of terms lead to *exponential equations.*

EXERCISES

1. Find the 10th term of the geometric progression 1, 2, 4, ⋯.

2. Find the 10th term of the geometric progression 1, − 2, 4, ⋯.

3. Find the 12th term of the geometric progression

$$\sqrt{3},\ -3,\ 3\sqrt{3},\ \cdots.$$

4. Find the 7th term of the geometric progression

$$2-\sqrt{3},\ 2\sqrt{3}-3,\ 6-3\sqrt{3},\ \cdots.$$

5. Find the nth term of the geometric progression

$$a,\ a(1+i),\ a(1+i)^2,\ \cdots.$$

6. Find the nth term of the geometric progression 1, $\frac{1}{2}$, $\frac{1}{4}$, ⋯.

7. Find the sum of 7 terms of the geometric progression

$$1+2+4+\cdots.$$

8. Find the sum of 7 terms of the geometric progression

$$1+\frac{1}{2}+\frac{1}{4}+\cdots.$$

9. Find the sum of n terms of the geometric progression

$$1+\frac{1}{2}+\frac{1}{4}+\cdots.$$

10. Find the sum of n terms of the geometric progression

$$a+a(1+i)+a(1+i)^2+\cdots.$$

11. Find the sum of n terms of the geometric progression

$$a+a(1+i)^{-1}+a(1+i)^{-2}+\cdots.$$

12. Insert 5 geometric means between 2 and 6.

13. Insert 5 geometric means betweens 2 and 3.

14. Given $a_1=5$, $a_6=15625$, $n=6$. Find r and s_6.

15. Given $a_1=5$, $s_6=19530$, $n=6$. Find r and a_6.

16. Given $a=\frac{1}{3}$, $r=\frac{3}{2}$, $n=n$. Find a_n.

17. Find the sum of n terms of the geometric progression

$$x-y+\frac{y^2}{x}-\frac{y^3}{x^2}+\cdots.$$

18. Given r, n, and s_n. Find a_n.

19. Given r, n, and a_n. Find s_n.

20. Given, a_1, r, and a_n. Find n.

21. Prove that if a set of numbers form a geometric progression, their kth powers form a geometric progression whose ratio is r^k.

22. The sum of three numbers in geometric progression is 221 and the last exceeds the first by 136. What are the numbers?

23. Find the sum of the progression

$$\left(r-\frac{1}{r}\right)^2+\left(r^2-\frac{1}{r^2}\right)^2+\left(r^3-\frac{1}{r^3}\right)^2+\cdots\left(r^n-\frac{1}{r^n}\right)^2.$$ [BOURLET.]

24. Prove that if a set of numbers form a geometric progression, their logarithms form an arithmetic progression.

25. Find the sum of n terms of the geometric progression

$$x^{n-1}-x^{n-2}y+x^{n-3}y^2+\cdots.$$

26. Find the sum of n terms of the geometric progression

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2\cdots.$$

27. From Exs. 26 and 27 deduce the theorems for the divisibility of

$$x^n+y^n \quad \text{and} \quad x^n-y^n.$$

130. Infinite Geometric Progressions. If an air pump whose cylinder contains 1 liter be attached to a vessel containing 9 liters, the first stroke of the piston will remove one tenth of the air, leaving nine tenths behind. The air left behind expands immediately to fill the whole space, so that the second stroke removes one tenth of the remainder, and so on. If the weight of the air at the beginning of the experiment be denoted by 1, the weights of air in the vessel at the beginning of each successive stroke will be

$$1, \quad .9, \quad .81, \quad .729, \quad .6561, \quad \cdots,$$

and the weights removed will be

$$.1, \quad .09, \quad .081, \quad .0729, \quad .06561, \quad \cdots.$$

Clearly, both of these sets of numbers are geometric progressions with ratios equal to .9. Theoretically, the total weight removed after the nth stroke will be the sum

$$s_n=.1+.09+.081+.0729+\cdots, \text{ to } n \text{ terms.}$$

By (6), § 128, this sum is

$$s_n = \frac{.1 \times (.9)^n - .1}{.9 - 1} = 1 - (.9)^n.$$

Suppose, now, that this process is carried on *indefinitely.* As n increases $(.9)^n$ becomes smaller and smaller, and by taking n large enough it can be made smaller than any previously assigned number. When this is done, the sum $1-(.9)^n$ will differ from 1 by a number less than any previously assigned number. We say that s_n, or $1-(.9)^n$, *approaches the limit* 1, and we write

$$\lim_{n=\infty} s_n = \lim_{n=\infty} [1-(.9)^n] = 1.$$

The physical interpretation of this process is, of course, that the weight of air in the vessel tends toward zero as the pumping is continued indefinitely.

A geometric progression like the one just discussed, in which the ratio is less than unity, and the number of terms is permitted to increase indefinitely, is called an ***infinite geometric progression.*** The notion is a very important one in both pure and applied mathematics.

In general, let

$$s_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$$

be a geometric progression in which $r < 1$.

By formula (6),

$$s_n = \frac{a_1 r^n - a_1}{r - 1} = \frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}.$$

Since r is a proper fraction, the powers of r approach zero as a limit. Consequently, $a_1 r^n/(1-r)$ approaches zero, and

$$\textbf{(7)} \qquad \lim_{n=\infty} s_n = \frac{a_1}{1-r}.$$

Formula (7) enables us to solve the air-pump problem directly; for, we have $a_1 = .1$ and $r = .9$, so that $\lim_{n=\infty} s_n$ becomes $.1/(1-.9) = 1$.

EXERCISES

1. Find the limit of the sums of each of the following infinite geometric progressions.

(a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots.$

(b) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots.$

(c) $1+\frac{2}{3}+\frac{4}{9}+\cdots.$

(d) $1+2-\sqrt{3}+7-4\sqrt{3}+\cdots.$

2. Find the common fraction which gives rise to the repeating decimal .555 ⋯, by finding the limit of the sum of the series

$$.5+.05+.005+\cdots.$$

Frame a rule by which the common fraction may be written down at once.

3. Find the common fractions from which the repeating decimals may be derived.

(a) .272727 ⋯. (b) .109109 ⋯. (c) .347347 ⋯. (d) .02323 ⋯.
(e) .52323 ⋯ = .5 + .02323 ⋯.

4. Find the limit of the sum of the series $1+x+x^2+x^3+\cdots$, when $x<1$.

5. If an elastic ball thrown upward to a height of 20 feet rebounds each time it strikes the ground to one third the height from which it fell, theoretically, how far will it travel?

6. A farm yields $1000 net per year. If this rate of return can be kept up indefinitely and money is worth 5%, what is the farm worth?

[Hint. The first annual return due one year hence is worth now 1000/1.05; the second $1000/(1.05)^2$; and so on.]

7. Suppose that instead of producing $1000 net each year the farm in Ex. 6 must *lie fallow* every other year, making a return in alternate years only. What would it be worth?

131. Means and Averages. The ***arithmetic mean*** or ***simple average*** of several numbers is the sum of the numbers divided by their number. In symbols, the arithmetic mean of the n numbers $a_1, a_2, \cdots, a_n$ is

$$A=\frac{a_1+a_2+\cdots+a_n}{n}. \tag{8}$$

The ***geometric mean*** of n numbers $a_1, a_2, \cdots a_n$ is the nth root of their product and may be written

$$G = \sqrt[n]{a_1 a_2 \cdots a_n}. \tag{9}$$

Closely connected with the arithmetic mean is the ***weighted average,*** which figures largely in problems in statistics. The weighted average of several numbers $a_1, a_2, \cdots, a_n$ is the sum of the products obtained by multiplying each number by a number called a ***weight,*** divided by the sum of the weights. It is given by

$$W = \frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n}. \tag{10}$$

The weighted average has an important physical meaning also, namely, if $w_1, w_2, \cdots, w_n$ be the weights of n masses lying in a line located at distances $a_1, a_2, \cdots, a_n$ respectively from a given point O, then W is the distance from O to the ***center of gravity*** of the masses whose weights are $w_1, w_2, \cdots, w_n$.

When a set of numbers form an arithmetic progression, their reciprocals form an ***harmonic progression,*** and the reciprocal of the arithmetic mean of two numbers is the ***harmonic mean*** of the reciprocals of the numbers. In finding averages great care must be taken to use the proper mean.

EXERCISES

1. Prove that any term of an arithmetic progression is the arithmetic mean of the two adjacent terms.

2. Prove that any term of a geometric progression is the geometric mean of the two adjacent terms.

3. Prove that the harmonic mean of a and b is

$$H = \frac{2ab}{a+b}. \tag{11}$$

4. Express the geometric mean of two numbers in terms of their arithmetic and harmonic means.

5. A line divided internally and externally so that the segments have the same ratio in both divisions, is said to be *divided harmonically*. Prove that when a line is divided harmonically, the whole line is the harmonic mean of the two segments.

[HINT. Let the line $AB = b$, and the segments $AP = a$ and $AP_1 = c$; then express b in terms of a and c.]

6. Prove that the roots of a quadratic equation $ax^2 + 2bx + c = 0$ are
(*a*) real when a, b, and c are in arithmetic progression;
(*b*) equal when a, b, and c are in geometric progression;
(*c*) imaginary when a, b, and c are in harmonic progression.

7. Prove that the logarithm of the geometric mean of two numbers is the arithmetic mean of the logarithms.

8. Three masses weighing 5, 7, and 11 pounds are located 2 feet to the right, 3 feet to the left, and 4 feet to the right, of a given point, respectively. Where is the center of gravity with respect to the point?

9. A student receives marks as follows: 85 in a 2-credit course, 90 in a 5-credit course, 93 in a 3-credit course, 91 in a 3-credit course, and 80 in a 3-credit course. What is his weighted average?

10. A community has 1 citizen with income of \$10,000, 10 with incomes of \$5000, 30 with incomes of \$3000, 50 with incomes of \$2000, 20 with incomes of \$1000. Find (*a*) the average income of citizens of the community; (*b*) the income of the average citizen of the community.

11. A merchant increased his capital the first year by 15%, the second by 20%, and the third by 25%. What was the average rate of increase for the three years?

[HINT. If r be the average rate of increase, the capital at the end of the three years would be the original capital multiplied by $(1 + r)^3$.]

12. Find the formula for the average rate of increase, over a period of n years when the annual rates are $r_1, r_2, r_3, \cdots, r_n$.

13. Two successive miles are traveled, the first at the rate of a miles per hour, and the second at the rate of b miles per hour. What is the average rate? What kind of a mean expresses the average rate?

14. Three successive miles are traveled at the rates of 3, 5, and 7 miles per hour, respectively. What is the average rate?

132. Compound Interest. When a sum is placed at ***compound interest***, the interest is added to the principal, or *converted into principal*, at stated intervals. The ***rate of interest*** is reckoned at a given per cent per year, though the conversion interval is frequently less than a year. The total amount due at the end of n years is called the ***compound amount.***

PROBLEM. *To find the compound amount on a principal A for n years at rate i.*

At the end of the first year the amount will be

$$A + Ai = A(1 + i).$$

At the end of the second year it will be

$$A + Ai + (A + Ai)\,i = A(1 + i)^2.$$

Similarly, at the end of n years the compound amount will be

$$A' = A(1 + i)^n. \tag{12}$$

For convenience of computation formula (12) is supposed to be true for fractional as well as for integral values of n.

The value of A' is easily found by logarithms when A, i, and n are given. Moreover, from formula (12) the value of any one of the numbers A, i, and n may be found where the other three numbers in the formula are known.

133. Present Value. The most important formula to be derived from (12) is that found by solving for A. We find

$$A = \frac{A'}{(1 + i)^n} = A'(1 + i)^{-n},$$

or, if we replace A and A' by the symbols V and P,

$$V = P(1 + i)^{-n}. \tag{13}$$

The quantity V is called the ***present value*** of the principal P due n years hence. It is the amount which, if placed at interest now, will amount to P in n years. In other words, it is the cash value of the obligation P due in n years without interest. The value of V like that of A' is easily found by

logarithms. However, the numbers $(1+i)^n$, the amount of unit principal, and $(1+i)^{-n}$, the present value of unit principal due in n years, occur so frequently that their values have been tabulated for a large number of rates, and for times up to 100 years or more. A table of compound amounts and present values is given in the tables at the back of this book (Tables D and E).

EXERCISES

1. To what sum will $ 1000 amount if put at compound interest at 4 % for 10 yrs.?

2. Find the amount of $ 1345.27 at 4 % compound interest for 10 years and 6 months.

[HINT. $(1.04)^{10\frac{1}{2}} = (1.04)^{10} \cdot (1.04)^{\frac{1}{2}}$.]

3. Find the compound amount of $ 1235.27 for 3 years at 6 %, compounded semiannually.

[HINT. The multiplier will be the same as the multiplier for 6 years at 3 % compounded annually.]

4. Find the compound amount of $ 2375.21 for 4 years at 4 %, compounded quarterly.

5. What sum put at compound interest for 5 years at 4 %, compounded annually, will amount to $ 1000?

6. What is the present value of $ 5000, due in 4 years without interest, when money is worth 6 %?

7. A man owes two sums, $ 2000 due without interest in 2 years, and $ 3000 due in 4 years without interest. He wishes to arrange to discharge both of these obligations in 3 years. What will be the amount required if money is worth 6 %?

8. In what time will a sum of money double itself if placed at compound interest at 5 %?

9. At what rate must a sum of money be invested at compound interest in order that it will double itself in 15 years?

10. Find the formula for the time when the amount, the principal, and the rate are unknown.

11. Find a formula for the rate when the amount, the principal, and the time are known.

12. In 1907 the consumption of coal in the United States was 480,363,000 tons, which was 7.36 % in excess of the consumption in 1906. Assuming the rate of increase to continue, what will be the consumption in 1927? (Use logarithms and give result in millions of tons.)

13. A common rule for the time in which any sum of money will double itself at compound interest is, "Divide .69 by the rate and add one third of a year." Prove that the rule is approximately correct.

[HINT. $\log_e 2 = .69$ and $\log_e (1 + i) = i - \frac{i^2}{2} + \frac{i^2}{3} - \cdots$].

14. At a certain university which had 4000 students in 1910, the increase is about 10 % each year over the previous year's attendance. Assuming the rate of increase to continue, what would the attendance be in 1920?

15. Construct a graph to show the compound amount of 1 dollar at 4 % as the time varies.

16. If money is worth 6 %, what is the cash price of a farm for which a man pays $1000 cash, $1240 at the end of one year, $1180 at the end of two years, $1120 at the end of three years, and $1060 at the end of four years? Compare result with the case of a man who buys a farm for $4000, agreeing to pay $1000 cash and $1000 per year with interest at 6 % on all sums remaining unpaid, and draw your conclusion as to the necessity for compound interest.

134. Annuities. An ***annuity***, or, to speak more accurately, an ***annuity certain***, is a series of equal payments made at equal intervals of time during a fixed term of years. The first payment is supposed to be made at the end of the first year. The elements that enter into an annuity are the amount of the periodic payment, the rate of interest, and the time, which we shall denote by a, i, and n, respectively. We shall speak of an *annuity of a per annum* when the sum of the payments made in one year amounts to a.

The ***amount of an annuity*** is the sum that would be due at the end of the term, if no payments were made but all were put at compound interest as soon as due.

The ***present value of an annuity*** of a per annum is the amount that one could afford to pay in cash for the privilege of receiving the payments in regular order.

PROBLEM I. *To find the amount of an annuity of a per annum.* The amount of annuity is the sum of the amounts of the annual payments, each accumulated to the end of the transaction. The first payment will be on interest for $n-1$ years, the second for $n-2$ years, and so on, the last payment being a cash payment. By the compound interest formula, the amounts of the payments are

$$a(1+i)^{n-1},\ a(1+i)^{n-2},\ \cdots,\ a(1+i),\ a,$$

and the amount of the annuity is the sum of these amounts. Written in reverse order, the amounts form a geometric progression with ratio $1+i$. Denoting the sum by S we have

$$S = a + a(1+i) + a(1+i) + \cdots + a(1+i)^{n-1}.$$

The value of this sum, as given by problem (10), § 129, is

$$S = a\frac{(1+i)^n - 1}{i}. \tag{14}$$

Tables for the amount of an annuity with a unit payment are given in textbooks on the mathematical theory of investment. The value of S is easy to find from the formula (14) if a table of compound interest is available.

PROBLEM II. *To find the present value of an annuity of a per annum.*

The present value of the annuity is the sum of the present values of the payments. The present value of the first payment is $a(1+i)^{-1}$, of the second $a(1+i)^{-2}$, and so on, to the last, whose present value is $a(1+i)^{-n}$. These numbers form a geometric progression whose ratio is $(1+i)^{-1}$. Denoting

the sum by V and making use of the results of Example 11, § 129, we have

$$V = a \cdot \frac{1-(1+i)^{-n}}{i}. \tag{15}$$

Like the values of S, the values of V for a unit payment have been tabulated, so that with a table at hand V is found by multiplying a by the appropriate number taken from the table.

EXERCISES

1. A man saves $100 annually which he deposits with a savings bank which pays 4 %. How much will he have saved at the end of 10 years?

2. In 1907 the coal consumption of the United States was 480,363,000 tons and the rate of increase over the preceding year was about 7 %. At that rate what would be the total consumption to the end of the year 1927, beginning at the first of the year 1908?

3. A man agrees to pay $1000 cash at the end of each year for five years for a house. If money is worth 6 %, what is the cash value of the house?

4. A mine will produce $10,000 net per year for 20 years and money is worth 6 %. Leaving aside extraordinary risks, what is the mine worth?

135. Sinking Fund and Amortization Problems. When a debt S, payable at some future time, is contracted, provision for payment should be made by setting aside each year a sum large enough so that the total accumulation will equal the debt when the latter falls due. The sums set aside each year constitute an *annuity that will amount to S*, and the fund into which the annual payments are made is called a ***sinking fund***. A fund of this kind may be accumulated for the purpose of replacing buildings or equipment that are wearing out. In this case the fund is called a ***depreciation fund***, and the annual payment is a *charge for depreciation*. The theory is identical for the two cases.

Let A_S denote the annuity that will amount to S. The value of A_S is found by replacing a by A_S in equation (14), § 134, and then solving for A_S. The required formula is

$$A_S = S\frac{i}{(1+i)^n - 1}. \tag{16}$$

Another important problem arises when the debtor agrees to pay *principal and interest* in equal annual installments. This form of payment is one form of ***amortization.*** Here again the annual payments constitute an annuity, which, in this case, has D the face value of the debt for its present value. The annual payment is spoken of as *the annuity that D will purchase.*

Let A_D denote the annuity that D will purchase. The value of A_D is found by replacing V by D and a by A_D in equation (15) and then solving the equation for A_D. The result is given by the equation

$$A_D = D\cdot\frac{i}{1-(1+i)^{-n}}. \tag{17}$$

The factor $i/[(1+i)^n - 1]$ occurring on the right of equation (16) may be found directly from a table for "the annuity that will amount to 1," while the factor $i/[1-(1+i)^{-n}]$ in (17) may be taken from a table for "the annuity that 1 will purchase."

Formulas (12), (13), (14), (15), (16), and (17) are the basis of the mathematical theory of investment.

EXERCISES

1. What sum must be set aside at the end of each year to repay a loan of $100,000 due in 20 years, if the payments set aside can be accumulated at 4 %?

2. A locomotive costing $75,000 wears out in 25 years. If money can be accumulated at 5 %, what should be the annual charge for depreciation?

3. A company borrows $100,000 for 20 years, agreeing to repay the loan, principal, and interest, at 6% in 20 equal annual installments. What is the yearly amortization payment?

4. A man buys a house for $6000, agreeing to pay $1000 down and the balance, principal, and interest at 6%, in 5 equal annual payments. What is the amount of the annual payment?

136. The Compound Interest Law. If a be any base, the consecutive integral powers of a may be arranged as in the following table.

Exponent $= x$	0	1	2	3	4	5	6	7	⋯
Power $= y$	1	a	a^2	a^3	a^4	a^5	a^6	a^7	⋯

The relation between an exponent x and its corresponding power y is given by the equation

$$(18) \qquad y = a^x.$$

It is clear that if the exponents form an arithmetic progression the corresponding powers will form a geometric progression, whether the exponents are integers or not. Since x is the logarithm of y to the base a, this fact may be stated as follows. *If a set of numbers are in geometric progression, their logarithms are in arithmetic progression, and conversely.*

The formula for compound interest is a special case of (18). If interest be converted into principal m times a year, the compound amount A at rate i for a principal A at the end of t years is given by the formula

$$(19) \qquad A' = A\left(1 + \frac{i}{m}\right)^{mt}.$$

Let $i/m = 1/u$, so that $m = ui$. With this notation

$$(20) \qquad A' = A\left[\left(1 + \frac{1}{u}\right)^{u}\right]^{it}.$$

Suppose, now, that the conversion interval becomes shorter and shorter, so that in the limit interest is converted into principal instantaneously. In this way we arrive at the notion of *instantaneous* or *continuous compound interest*, a thing that is actually approximated in the business of large concerns which receive interest payments many times each day. However, the great value of the notion is due to its application to physical rather than to financial problems.

To find the formula for instantaneous compound interest it is necessary for m, and consequently for u, to increase indefinitely. In books on analysis it is shown that

$$(21) \qquad \lim_{u=\infty}\left(1+\frac{1}{u}\right)^u = 2.718218 \cdots.$$

This limit is denoted by the letter e.* If y denote the limit toward which A' approaches, formula (20) becomes

$$(22) \qquad y = Ae^{it}, \ (e = 2.718218 \ \cdots\ ).$$

With the value of e known approximately, the amount at instantaneous compound interest is easily found.

To realize more clearly the meaning of equation (22), it is important to note that if A be the principal at the beginning of any conversion interval, the interest for the interval is iA/m, and the amount at the end of the interval is $A + iA/m$. In other words, the interest, or *increment per interval*, is *proportional to the principal*. Equation (22) may therefore be looked upon as a formula giving the *law of growth* of any magnitude whose *rate of increase at every instant is proportional to the quantity at the instant*. This law, which is followed in the case of many natural phenomena, has been named by LORD KELVIN the ***Compound Interest Law.***

* The number e, which is one of the most important in the whole field of analysis, is the base of the so-called Napierian system of logarithms.

If the rate i is negative, the change per interval in A, namely, iA/m, is a *decrement*, or, in financial matters, a *discount*. In all cases the constant A is the measure of the magnitude at the time $t = 0$. Why?

Among the many applications of the compound interest law the following may be mentioned:

(1) By Newton's law of cooling the temperature at time t is given by the formula

$$\theta = \theta_0 e^{bt},$$

where θ_0 is the initial temperature, and b is a constant rate to be determined for each substance.

(2) The number of bacteria per cubic centimeter at the time t in the presence of unlimited food, and under proper temperature conditions, is given by

$$N = N_0 e^{kt},$$

where N_0 is the initial number and k is the rate of increase.

(3) The difference in potential at time t of the two coatings of an electrical condenser of capacity k, discharging through resistance R, is

$$v = v_0 e^{\frac{-1}{kR}t},$$

where v_0 is the potential at the beginning of the discharge.

EXERCISES

1. Find the instantaneous compound interest on $1000 for 3 years at 5 % and compare the result with the ordinary compound interest convertible annually, for the same time and at the same rate.

2. Radium is dissipated according to the law

$$q = q_0 e^{-kt},$$

where k is the rate of dissipation, q_0 the initial quantity, and q the quantity at time t. Find k on the hypothesis that one half the quantity disappears in 1800 years. How much is left at the end of 100 years?

3. The amount of light that passes through incompletely transparent material of thickness t is

$$L = L_0 e^{-kt},$$

where L_0 is the intensity of the beam striking the material, and k is a constant rate depending upon the material. Find k if 98 % of the light is transmitted through 1 cm. of glass.

4. In a certain culture 100 thousand bacteria were present at time $t = 0$ and 900 thousand at time $t = 10$. Find the rate of increase k. Find an expression for the number present at any time t. Use formula $N = N_0 e^{kt}$.

CHAPTER XV

PERMUTATIONS AND COMBINATIONS

137. Definitions and Fundamental Theorem. Consider a finite number of things which, for the moment, may be denoted by the letters $a_1, a_2, a_3, \cdots, a_n$.

In general, any r of these n things may be arranged in several different orders. Each order, or arrangement, of a set of r things taken from the set of n things is called a ***permutation*** of the r things. As a special case, r may be equal to n. Thus, the three letters a, b, c, give rise to the six permutations,

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba,$$

when taken three at a time; to the six permutations

$$ab, \quad ba, \quad ac, \quad ca, \quad bc, \quad cb,$$

when taken two at a time; and to the three permutations,

$$a, \quad b, \quad c,$$

when taken one at a time.

A set of r things considered without regard to order is called a ***combination***. The three letters, a, b, c, give rise to one combination, abc, of three letters, three combinations ab, bc, ca, of two letters, and three combinations a, b, c, of one letter.

138. Fundamental Theorem. *If a specified act can be performed in p ways and a second independent specified act can be performed in q ways, the two acts can be performed in combination in pq ways.*

The theorem is nearly self-evident. To fix ideas, let us suppose that the first act is passing out of one room having p openings into a hall, and the second passing from the hall into another room, having q openings (Fig. 37). Using the first

opening out of the first room, the passage can be made in q ways. This can be repeated for *each* of the p openings of the

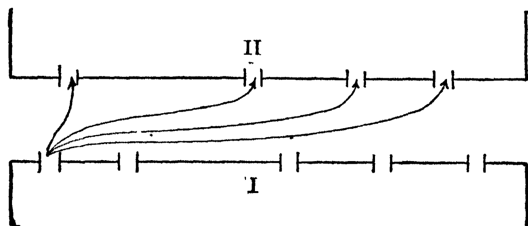


FIG. 37.

first room. Therefore, there are pq ways of going out of the first room and into the second.

139. Formulas for Permutations. PROBLEM I. *To find the number of permutations of n things taken all at a time.*

The number of permutations of n things taken all at a time is denoted by P_n. It is easy to see that $P_1 = 1$, $P_2 = 2 \cdot 1$, $P_3 = 3 \cdot 2 \cdot 1$. From these simpler cases it is reasonable to guess that $P_n = n(n-1) \cdots 3 \cdot 2 \cdot 1$, or, with the briefer notation

$$P_n = n! \tag{1}$$

The truth of this statement is easily proven by mathematical induction. For, suppose that (1) is true for $n = k$. Then, by hypothesis $P_k = k!$ Suppose that a $(k+1)$ st thing be added so that there are $k+1$ things, any one of which may occupy the first place. In order to form all possible permutations, there are two acts to be performed; first, the placing of one of the $k+1$ things first, second, the arranging of the remaining k things. The first act may be performed in $k+1$ ways and the second in $k!$ ways. By the fundamental theorem the number of ways of performing the two acts is $(k+1)k!$ But $(k+1) \times k! = (k+1)!$ Therefore, if (1) is true when $n = k$, it is true when $n = k+1$, and the induction is complete.

PROBLEM II. *To find the number of permutations of n things taken r at a time.*

Let ${}_nP_r$ denote the required number. For $r = 2$, the first thing may be arranged in n ways and the second in $n - 1$ ways. Therefore, by the fundamental theorem

$$ {}_nP_2 = n(n-1), $$

which is a special case of the more general formula

$$ (2) \qquad {}_nP_r = n(n-1)(n-2)\cdots(n-\overline{r-1}). $$

To prove that (2) is true for any integral value of r, which is not greater than n, suppose it to be true for $r = k$. Then by hypothesis,

$$ {}_nP_k = n(n-1)(n-2)\cdots(n-\overline{k-1}). $$

If the things be taken $k + 1$ at a time the required permutations may be formed by two acts. The first of these two acts is the arrangement of k things and the second is the selection of the $(k + 1)$st thing from the remaining $n - k$ things. By hypothesis the first act may be performed in

$$ n(n-1)(n-2)\cdots(n-\overline{k-1}) $$

ways. Since the second may be performed in $n - k$ ways, the two acts may therefore be performed in

$$ n(n-1)(n-2)\cdots(n-\overline{k-1})\times(n-k) $$

ways. Consequently,

$$ {}_nP_{k+1} = n(n-1)(n-2)\cdots(n-\overline{k-1})(n-\overline{(k+1)-1}). $$

This formula is exactly formula (2) with r replaced by $k + 1$. If, therefore, formula (2) is true for $r = k$, it is true for $r = k + 1$, and by complete induction, for any value of r.

If numerator and denominator of (2) be multiplied by $(n - r)!$ the numerator becomes $n!$, since $n - r$ is the next integer below $n - r + 1$. Consequently, (2) takes the useful form

$$ (3) \qquad {}_nP_r = \frac{n!}{(n-r)!}. $$

EXERCISES

1. There are five routes by which a passenger may go by rail from Madison to Chicago, and six from Chicago to Saint Louis. In how many ways may the trip from Madison to Saint Louis by way of Chicago be made?

2. How many five figure numbers can be written by means of 5 digits if no digit is repeated in any number?

3. Every term of a determinant of the 6th order is of the form $a_x b_y c_z d_u e_v f_w$ where the subscripts x, y, z, u, v, w, are the numbers 1, 2, 3, 4, 5, 6, in some order. How many terms does the expansion of the determinant contain?

4. A signal corps has 6 flags to be displayed in a horizontal row three at a time. How many signals can be made?

5. How many numbers can be written with six digits if no number contains the same digit twice?

6. How many signals are possible with six flags of different colors displayed in a line?

140. Formulas for Combinations. PROBLEM III. *To find the number of combinations of n things taken r at a time.*

Let the number required be denoted by ${}_nC_r$. Consider a single combination of r things. The things of this combination may be arranged in $r!$ ways according to (1), § 139. Moreover, this statement is true of every one of the ${}_nC_r$ combinations, and the totality of permutations thus obtained is the number of permutations of n elements taken r at a time. Consequently,

$$ {}_nC_r \times r! = {}_nP_r = \frac{n!}{(n-r)!}. $$

From this equation we obtain

$$ {}_nC_r = \frac{n!}{r!(n-r)!}. \tag{4} $$

EXERCISES

1. A society has 30 members. In how many ways may a committee of 3 members be chosen?

2. How many different products, each containing four factors, may be constructed from 8 prime numbers?

3. How many numbers between 100 and 200 may be written with the digits 1, 2, 3, 4, and 5, if no digit is used twice in any number?

141. The Binomial Theorem. The theory of combinations furnishes an elegant method for the proof of the binomial theorem when the exponent is a positive integer.

The expression $(x + a)^n$ may be considered as the product of n factors

$$x + a_1, \quad x + a_2, \quad x + a_3, \quad \cdots, \quad x + a_n,$$

when the a's all become equal to a. By the generalized distributive law (§ 11),

$$(5) \quad (x + a_1)(x + a_2) \cdots (x + a_n) \equiv x^n + (a_1 + a_2 + \cdots + a_n)x^{n-1} + (a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n)x^{n-2} + (a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n)x^{n-3} + \cdots + a_1a_2a_3 \cdots a_n,$$

where the coefficients of x^{n-1}, x^{n-2}, x^{n-3}, $\cdots$ are the sums of all possible products of the a's taken one at a time, two at a time, and so on. If $a_1 = a_2 = a_3 = \cdots = a_n = a$, (5) may be written in the form

$$(6) \quad (x + a)^n \equiv x^n + C_1ax^{n-1} + C_2a^2x^{n-2} + C_3a^3x^{n-3} + \cdots + a^n,$$

where C_1, C_2, C_3, $\cdots$ are integers which represent the number of terms in the coefficients of x^{n-1}, x^{n-2}, $\cdots$ respectively. But a closer examination of (5) shows that C_1 is the number of combinations of the a's taken one at a time, C_2 the number of combinations taken two at a time, and so on; that is,

$$C_1 = {}_nC_1, \quad C_2 = {}_nC_2, \quad C_3 = {}_nC_3, \quad \cdots, \quad C_r = {}_nC_r \cdots.$$

Hence the binomial theorem may be written in the form

$$(7)\qquad (x+a)^n = x^n + {}_nC_1\,ax^{n-1} + {}_nC_2\,a^2x^{n-2} + {}_nC_3\,a^3x^{n-3} + \cdots + {}_nC_r\,a^rx^{n-r} + \cdots + a^n.$$

That this form for the binomial theorem is identical with that given in § 118, is easily seen by writing out the values of the binomial coefficients,

$$(8)\qquad {}_nC_1,\quad {}_nC_2,\quad {}_nC_3,\quad \cdots {}_nC_r \cdots.$$

The coefficient of the rth term is, clearly,

$$(9)\qquad {}_nC_{r-1} = \frac{n!}{(r-1)!\,(n-r+1)!}.$$

MISCELLANEOUS EXERCISES

1. In a baseball nine one man can catch and one can pitch. In how many ways can the team be played?

2. In how many ways can a committee composed of two Germans and three Englishmen, be chosen from 10 Germans and 15 Englishmen?

3. How many lines can be drawn through 15 points, each line to pass through two points, if no three points lie on a line?

4. If no three of a group of 21 points lie on a line and no four in a plane, how many planes are determined by the group?

5. Prove by the theory of combinations that the binomial coefficients equidistant from the ends are equal.

6. If n things lie on the circumference of a circle (or on any closed curve) a given arrangement is called a *circular permutation.* Prove that the number of circular permutations of n things is $(n-1)!$

7. In how many ways may the letters of the word *algebra* be arranged?

[Hint. Note the repetition of one letter.]

8. Find a formula for the number of permutations of n things taken all at a time when r of them are alike.

9. Find a formula for the number of permutations of n letters taken all at a time, if r of the letters are a's, and s are b's.

10. Write down the coefficient of $a^rb^sc^t$ in $(a+b+c)^n$.

CHAPTER XVI

PROBABILITY

142. Simple Probability. If a bag contains 5 white and 3 black balls and a ball be drawn at random, any one of the 8 balls may be drawn. This fact may be expressed by saying that any one of the possibilities is equally likely to happen. Of the 8, any one of 5 would result in drawing a white ball, and any one of 3, in drawing a black ball. Common sense would seem to point out that 5 possibilities or 5 of the 8 cases are favorable to drawing a white ball and 3 unfavorable. Upon this basis LAPLACE formulated the definition of ***simple probability*** as follows: *The probability that, among several equally likely events, an event will happen in a certain way, is the ratio of the number of favorable cases to the total number of cases.*

In the example of the balls the probability of drawing a white ball is, according to the definition, 5/8 and the probability of drawing a black ball is 3/8. In the general case, if p represents the probability, a the number of favorable cases, and b the number of unfavorable cases,

$$(1) \qquad p = \frac{a}{d+b}.$$

The definition of probability, though it may be said to be based on common sense, is, like all definitions, purely arbitrary. It does not mean that if the drawing were to be performed 8 times, 5 drawings would bring white balls and 3 drawings would bring black balls, but, rather, that if a great number of drawings were to be made under exactly the same

conditions, the ratio of white balls drawn to black balls drawn, would be as 5 to 3.

If the cases are all favorable, the event is *certain* and

(2) $$p = \frac{a}{a+0} = 1;$$

while if no case is favorable the event is *impossible* and

(3) $$p = \frac{0}{a+b} = 0.$$

Consequently, *certainty* is expressed by 1, and *impossibility* by 0.

THEOREM. *If p is the probability that an event will happen and q the probability that it will fail,*

(4) $$p + q = 1.$$

For, $p = a/(a+b)$ and $q = b/(a+b)$. Therefore

$$p + q = \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

EXERCISES

1. A single cubical die with faces numbered from 1 to 6 is thrown once. What is the probability that the face numbered 4 will lie uppermost?

SOLUTION. One case is favorable and 5 are unfavorable. The probability required is therefore 1/6.

2. Two coins are thrown into the air simultaneously (or in succession). What is the probability that both will fall "heads"?

[HINT. They may fall in only four possible ways; namely, both heads, A heads and B tails, A tails and B heads, both tails.]

3. A die is thrown once. What is the probability that the number of points uppermost shall be less than 4?

4. Ten balls numbered from 1 to 10 are placed in a bag and two drawn at random. What is the probability that the two are numbered 4 and 7?

[HINT. How many ways are there of selecting 2 out of 10 things?]

5. What is the probability that 12 coins tossed into the air at the same time will all fall heads? What is the probability that a single coin tossed into the air 12 times will fall heads every time?

[HINT. In how many ways may 12 coins fall?]

6. Two balls are drawn simultaneously from a bag containing 4 white and 6 black balls. What is the probability that both balls will be white? That both will be black. That one will be black and one white?

7. Statistics of the insurance companies show that of 100,000 children living at age 10, 740 will die within a year. What is the probability that a child 10 years of age will die within a year? That it will survive for one year?

8. Of 100,000 persons living at age 10, 69,804 are supposed to be alive at age 50. What is the probability that a boy aged 10 will die before he is 50?

143. Partial and Total Probability. An event may happen in any one of several series of mutually exclusive events. The probability that it will happen in a given series is called ***partial probability.*** The probability that it will happen without regard to any series is called ***total probability.*** To illustrate, suppose a bag contains 10 white balls of which 6 are marked with crosses and 4 not; 8 black balls, 5 with crosses and 3 without; and 7 yellow balls, 4 with crosses and 3 without. The probability of drawing a ball of any given color and marked with a cross is partial probability and the probability of drawing a ball with a cross, but without regard to color, is total probability.

In the example just given there are three partial probabilities, as follows: The probability of drawing a white ball with a cross is $6/25$; that of drawing a black ball with a cross is $5/25$; and that of drawing a yellow ball with a cross is $4/25$. The total probability, that is, the probability of drawing a ball with a cross without regard to color, is $15/25$, since 15 out of 25 balls have crosses. The important fact is that

the sum of the partial probabilities is exactly equal to the total probability.

THEOREM. *The total probability of an event is the sum of its partial probabilities.*

Let m be the number of possible cases, and let $a_1, a_2, \cdots, a_n$ be the number of favorable cases in each one of n series of events. Further, let p be the total probability, and $p_1, p_2, \cdots, p_n$ the partial probabilities.

By the definition of total probability,

$$p = \frac{a_1 + a_2 + \cdots a_n}{m} = \frac{a_1}{m} + \frac{a_2}{m} + \cdots + \frac{a_n}{m}.$$

But the fractions $\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_n}{m}$ are the partial probabilities. Consequently,

$$p = p_1 + p_2 + p_3 + \cdots + p_n, \tag{5}$$

as was to be proved.

144. Compound Probability. If the happening of either one of two events has no influence upon the happening of the other, the two events are said to be *independent*. The probability that two independent events shall happen simultaneously (or in succession) is called ***compound probability.*** For example, if two coins are thrown into the air, or if two throws are made with one coin, the probability that both falls will give heads is compound.

THEOREM. *The compound probability of two events is the product of their simple probabilities.*

Let $p_1 = a_1/m_1$ and $p_2 = a_2/m_2$ be the simple probability, and p the compound probability. The two events may happen through any one of the a_1 favorable cases for the first and any one of the a_2 favorable cases for the second. By the fundamental theorem for permutations and combinations (§ 138), the number of favorable cases for the happening of

the two events is $a_1 a_2$. In the same way the number of possible cases is $m_1 m_2$. Consequently,

$$p = \frac{a_1 a_2}{m_1 m_2} = \frac{a_1}{m_1} \times \frac{a_2}{m_2},$$

or,

$$(6) \qquad p = p_1 \times p_2,$$

as was to be proven.

COROLLARY I. *The probability that several independent events whose simple probabilities are $p_1, p_2, \cdots, p_n$, will happen simultaneously (or in succession), is*

$$(7) \qquad p = p_1 p_2 \cdots p_n.$$

COROLLARY II. *The probability p that n events whose simple probabilities are $p_1, p_2, \cdots, p_n$ shall all fail is*

$$(8) \qquad 1 - p = (1 - p_1)(1 - p_2) \cdots (1 - p_n);$$

the probability that the first r of them shall happen and the remainder fail is

$$(9) \qquad p_1 p_2 \cdots p_r (1 - p_{r+1}) \cdots (1 - p_n).$$

EXERCISES

1. If the probability that A will win a race is $\frac{1}{3}$ and that B will win it is $\frac{1}{4}$, what is the probability that either A or B will win it?

2. What is the probability that a single throw of two dice will be less than 3?

3. The probability that A working alone can solve a problem is $\frac{1}{4}$ and that B working alone can solve it is $\frac{1}{3}$. What is the probability that the problem will be solved if both work at it, each alone?

[HINT. The problem will be solved if (*a*) A succeeds and B fails; (*b*) A fails and B succeeds; (*c*) both succeed.]

4. If p_1 represent the probability that a man will die within a year and p_2 the probability that his wife will die within a year, what are the probabilities (*a*) that the man will die and his wife survive? (*b*) that the man will survive and the wife will die? (*c*) that both will die? (*d*) that both will live?

145. The Mortality Table. One of the most important applications of the theory of probabilities is the application to certain problems relating to the duration of human life. The problems confronting the life insurance company, or the employer, either public or private, who wishes data for a pension system, or the judge who seeks to determine a life interest in an estate, all depend for their solution upon the theory of probability.

The instrument actually employed in the solution of all these problems is a ***mortality table***. A mortality table in its simplest form is a record showing how many persons out of a large number, all of the same age, die during each successive year until all are dead. Such a table is based upon observation and not upon mathematical considerations. Mortality tables differ for different countries, for different classes living side by side in the same country, and for persons of the same class engaged in different occupations, etc.

The table given in this book (Table F, p. 258) is called the ***American Experience Table*** and was compiled from the results of the experience of thirty American insurance companies to the end of the year 1874. This table is the basis upon which most of the life insurance business of this country is conducted. The methods actually employed in constructing a mortality table from observed data need not concern us here. In the above table the column headed x gives the age, the column headed l_x the number living at age x out of the 100,000 alive at the age of 10 years, and the column headed d_x the number dying between the ages x and $x+1$.

The probability that a person aged x years will live to age $x+1$ years is assumed to be

$$\frac{l_{x+1}}{l_x},$$

and the probability that he will die within the year is assumed to be

$$\frac{d_x}{l_x}, \quad \text{or} \quad \frac{l_{x+1} - l_x}{l_x}.$$

EXERCISES

1. What is the probability that a man aged 30 will be alive ten years later?

2. A man is 45 years of age and his son is 15. What is the probability that both will live 10 years?

3. A man and his wife are 30 and 28 years old respectively when their first child is born. What is the probability that both will be alive on the child's twenty-first birthday? That one will be alive on that date? That *only* one will be alive?

4. Find a formula for the probability that two persons of ages x and y will both live n years.

5. Find a formula for the probability that a man of age x will live n years and another of age y will die within n years.

6. From the Mortality Table plot the curve which shows the probability of dying for each year from ages 10 to the end of the table.

146. Mathematical Expectation. The value of a ***mathematical expectation*** is the product of a sum of money to be paid upon the happening of a specified event by the probability that the event will happen. For example, an insurance company agrees with a man 30 years of age to pay $ 1000 to his heirs in case of his death within one year. By the mortality table the probability that a man of 30 will die within a year is .00843. Consequently the value of the mathematical expectation is $ 8.43 at the end of the year, and if money is worth 3 % the present value of the expectation is $ 8.43 ÷ 1.03 = $ 8.19. This is the simplest form in which a problem in life insurance can present itself. For example, a life insurance company could afford to insure the lives of 10,000 men of age 30 for one year for $ 81,900 cash plus the cost of conducting the business.

EXERCISES

1. A stake of $10 is made contingent upon a throw of dice being less than 4. What is the mathematical expectation of the player?

2. If 9997 ships out of 10,000 of a given class and in a given condition reach port safely what would be the cost of insuring a ship with its cargo worth $500,000 for a single voyage, if interest, expenses, and profits are neglected?

3. What would be the cost of insuring a house worth $5000 for a year if two houses out of 1000 in its class burn down each year?

4. On the basis of the mortality table what would be the minimum cost of insuring the life of a man of age 35 for one year, neglecting interest and expenses?

CHAPTER XVII

SEQUENCES AND LIMITS

147. Definitions. A succession of numbers formed according to some definite law is called a ***sequence*** (§ 30). The set of numbers

$$.3, \quad .33, \quad .333, \quad .3333, \cdots,$$

which present themselves when we attempt to express the fraction 1/3 in the decimal notation is a sequence.

If we attempt to find an expression for the square root of 2 we are led successively to the numbers of the sequence

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \cdots.$$

The law of formation in this case is the rule for extracting the numerical square root.

The process of finding the ratio of the circumference to the diameter of a circle leads to the sequence

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \cdots.$$

If P_r denote the perimeter of a regular polygon with r sides inscribed in a circle with unit diameter, the sequence

$$P_3, \quad P_6, \quad P_{12}, \quad P_{24}, \cdots$$

is the sequence of numbers frequently used in finding the ratio of the circumference to the diameter.

In all these sequences the process of finding additional terms can be carried on indefinitely.

A variable may take for its values the numbers of a sequence. In such a case we say the variable *runs through* the numbers of the sequence. For example, let s_n be the variable which de-

notes the sum of n terms of the geometric progression

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots.$$

Then s_n runs through the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots,$$

as n increases.

148. Limits. In every example that has been given the sequence has the property that as the number of terms is increased, the successive terms approach nearer and nearer to a fixed number. Thus, the numbers of the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$$

approach nearer and nearer to 1, and if a sufficient number of terms are taken the difference between 1 and the terms of the sequence becomes indefinitely small. In such cases we say of both the sequence and the variable which runs through the terms of the sequence, that they approach a ***limit***. The sequence just given and the variable s_n which runs through the values of the sequence approach the limit 1. The sequence of perimeters approaches the circumference of the circumscribed circle as a limit.

The complete definition of the limit of a variable corresponding to such a sequence, is as follows.

If a variable x runs through the numbers of a sequence

$$x_1, \ x_2, \ x_3, \ x_4, \cdots, x_n, \cdots,$$

and if a constant number a exists, having the property that corresponding to any arbitrarily chosen positive number δ, a positive integer n may be found such that every one of the absolute values

$$(1) \qquad |a-x_n|, \ |a-x_{n+1}|, \ |a-x_{n+2}|, \cdots$$

is less than δ, the constant number a is called the ***limit*** *of the variable x.*

The relation between the variable x and its limit a is expressed by writing

(2) $$\lim_{n=\infty} x_n = a;$$

or, simply

(3) $$\lim_{n=\infty} x = a.$$

Formulas (2) and (3) are read *the limit of x_n (or of x)*, as n becomes infinite, *is equal to a*. A sequence whose numbers approach a limit is said to be *convergent*.

Any complete discussion of the theory of limits must be based upon a criterion for the existence of a limit. The following criterion, first given by CAUCHY, has been called by DU BOIS REYMOND, the ***General Principle of Convergence:*** *The necessary and sufficient condition that a sequence of real numbers, $x_1, x_2, x_3, x_4, \cdots$, may have a limit, is that for any arbitrarily chosen positive number ϵ, a value of n may be found such that every one of the absolute values*

(4) $$|x_n - x_{n+1}|,\ |x_n - x_{n+2}|,\ |x_n - x_{n+3}|, \cdots$$

shall be less than ϵ.

This general principle is frequently stated more briefly by saying that the sequence is convergent if a value of n can be formed such that

(5) $$|x_n - x_{n+p}| < \epsilon$$

for all positive integral values of p.

The general principle is here assumed to be true. Proofs may be found in textbooks on analysis.

EXAMPLE. Suppose s_n denotes the variable sum of n terms of the geometric progression

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots,$$

as n increases. Then s_n runs through the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$$

as n takes the values 1, 2, 3, ... Moreover, according to § 128, s_n is given by the formula

$$s_n = 1 - \frac{1}{2^n}.$$

To show the existence of a limit it is necessary to consider the absolute value

$$|s_n - s_{n+p}| = \left|\frac{1}{2^n} - \frac{1}{2^{n+p}}\right| = \frac{1}{2^n}\left(1 - \frac{1}{2^p}\right).$$

It follows that this absolute value can be made less than a given number if n can be found to satisfy the inequality

$$\text{(6)} \qquad \frac{1}{2^n}\left(1 - \frac{1}{2^p}\right) < \epsilon, \text{ or } 2^n > \frac{1}{\epsilon}\left(1 - \frac{1}{2^p}\right).$$

When logarithms are taken of both sides of the inequality written in the last form, we obtain the inequality

$$\text{(7)} \qquad n > \frac{\log\frac{1}{\epsilon} + \log\left(1 - \frac{1}{2^p}\right)}{\log 2}.$$

The number $1 - 1/2^p$ is less than unity for every admissible value of p, and its logarithm is negative. Consequently it will take a *larger* value of n to satisfy the inequality

$$\text{(8)} \qquad n > \frac{\log\frac{1}{\epsilon}}{\log 2}, \quad \text{or} \quad n > \frac{\log\frac{1}{\epsilon}}{.30103},$$

than it takes to satisfy the original inequality. It is easy to find values of n to satisfy the inequality (8), whatever ϵ may be. For example, if $\epsilon = .0000001$, $\log 1/\epsilon = 6$ and the inequality (8) is satisfied by $n = 20$ or any larger positive integer. The existence for a limit of the sequence is therefore proven by formula (8).

In a similar manner it may be shown that the limit of the sequence, and consequently of the variable s_n is 1.

149. The Limiting Values of Functions. In applications of the theory of limits, the problem usually presents itself in

the form of an inquiry as to the value toward which a function tends as the independent variable approaches a definite limit, or increases, indefinitely. The following examples will make the meaning clear.

EXAMPLE 1. In the problem of finding the sum of the geometric progression

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots,$$

as the number of terms is increased indefinitely, the sum of n terms is given by

$$s_n = 1 - \frac{1}{2^n}.$$

Here s_n is a function of n and the problem is reduced to finding

$$\lim_{n=\infty} s_n = \lim_{n=\infty}\left(1 - \frac{1}{2^n}\right).$$

EXAMPLE 2. If a rectangle of base x be inscribed in a semicircle of radius r, the rectangle is

$$A = x\sqrt{r^2 - \frac{x^2}{4}} \text{ (Fig. 38)}.$$

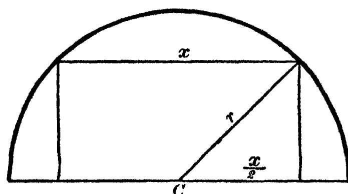


FIG. 38.

In a later problem (Problem 4, § 157) it is shown that the maximum area of this rectangle is found to be the limit of A as x approaches $r\sqrt{2}$. That is to say,

$$\text{maximum value of } A = \lim_{x \doteq r\sqrt{2}} x\sqrt{r^2 - \frac{x^2}{4}}.$$

In the second example we are not, properly speaking, dealing with a sequence, for x does not take a definitely determined sequence of values; but may take *every* value between two given values that lie between $x = 0$ and $x = r\sqrt{2}$. In such a case we say that the function $f(x)$ *approaches the limit A, as x*

approaches a, if, and only if, $\lim f(x) = A$ *for every sequence of values of x which approaches a.*

150. The Limit of an Algebraic Function. An ***algebraic function*** of x is a function which is formed from x by means of a finite number of the operations addition, subtraction, multiplication, division, and root extraction.

In more extended treatments it is proven that for any two functions which approach limits, the limit of their sum, the limit of their product, the limit of their quotient, are equal respectively to the sum of their limits, the product of their limits, and the quotient of their limits, provided that in the case of the quotient, the limit of the denominator is not zero. It is also proven that the limit of a power of a variable is equal to the same power of the limit of the variable.

These theorems may be summed up in a single easily remembered statement, as follows.

If $f(x)$ denote an algebraic function of a variable x, then

$$\lim_{x \doteq a} f(x) = f(a), \tag{9}$$

for all values of a for which the limit of the denominator is different from zero.

EXAMPLE. Let

$$f(x) \equiv \frac{2x^3 + 3x - 5x^{1/2}}{25 - x^2},$$

and suppose x approaches the constant a; then

$$\lim_{x \doteq a} f(x) = \lim_{x \doteq a} \frac{2x^3 + 3x - 5x^{1/2}}{25 - x^2} = \frac{2a^3 + 3a - 5a^{1/2}}{25 - a^2},$$

provided $|a| \neq 5$.

151. Infinity. Emphasis has been laid upon the fact that limits are finite and definite numbers. It frequently happens, however, that functions must be considered for which no finite limit exists, but for which the values increase beyond all

bounds as the independent variable approaches a particular value. (See § 54.) The simplest function of this sort is the function

$$f(x) \equiv \frac{1}{x},$$

which increases indefinitely as x approaches zero. In such cases it is customary to say that the function approaches the *improper limit, infinity*, or that the function *becomes infinite.* This circumstance is indicated by the notation

$$(10) \qquad \lim_{x \doteq 0} \frac{1}{x} = \infty.$$

The improper limit is said to be $+\infty$ or $-\infty$ according as the function increases numerically through positive or through negative values.

For example, if x runs through the sequence of values

$$1, .1, .01, .001, .0001, \cdots,$$

the function $1/x$ runs through the sequence

$$1, 10, 100, 1000, 10{,}000, \cdots,$$

which approaches the improper limit $+\infty$. For the same sequence of values for x the function $-1/x$ runs through the sequence

$$-1, -10, -100, -1000, -10{,}000, \cdots,$$

which approaches the improper limit $-\infty$.

An important example of the limit of a non-algebraic function is $\lim_{x \doteq 0} \log x$. When x runs through the sequence of values

$$1, .1, .01, .001, .0001, \cdots,$$

$\log x$ runs through the sequence of values

$$0, -1, -2, -3, -4, \cdots.$$

Clearly, as x approaches zero, $\log x$ approaches $-\infty$.

Note. The student should note that ∞ is not a number in the sense that it is an entity which obeys the ordinary rules of reckoning. For example, we may write $\infty + c = \infty$ where c is any number, zero or otherwise.

152. The Indeterminate Form 0/0. A fractional function

$$\frac{\phi(x)}{f(x)}$$

whose numerator and denominator are both zero for a given value of x is said to be ***indeterminate***, or to take the indeterminate form 0/0, for that particular value of x.

Up to the present time it has been necessary to say that a fraction is not defined for values of x which make it indeterminate. Nevertheless, the theory of limits furnishes a means of defining such functions for values for which they become indeterminate.

EXAMPLE 1. The fraction

$$f(x) \equiv \frac{1 - x^3}{1 - x}$$

becomes indeterminate for $x = 1$, although it has a definite, finite value for every other value of x, no matter how little that value may differ from 1.

Suppose that in the function given above, x runs through the sequence

$$0,\ .9,\ .99,\ .999,\ .9999,\ \cdots,$$

as it approaches 1. The function will then run through the sequence

$$1,\ 2.71,\ 2.9701,\ 2.997001,\ 2.99970001,\ \cdots,$$

which *apparently* approaches the limit 3. That the sequence actually does approach a limit may be proven rigorously. It is convenient, therefore, to *define* the fraction $(1 - x^3)/(1 - x)$, for the value $x = 1$, as

$$\lim_{x \doteq 1} \frac{1 - x^3}{1 - x}.$$

In general, if the fraction $\phi(x)/f(x)$ becomes indeterminate when $x = a$, it is defined for this value by the limit

$$\lim_{x \doteq a} \frac{\phi(x)}{f(x)},$$

provided, of course, the limit exists.

When an algebraic function becomes indeterminate for a given value of the variable by reason of a factor that occurs in

both the numerator and the denominator, the indeterminate form may be *evaluated* by removing the factor.

For example, the fraction of Ex. 1 may be written in the form

$$\frac{1-x^3}{1-x} \equiv 1 + x + x^2.$$

Since the two members of the equality are identically equal, their limits are equal.

Consequently,

$$\lim_{x \doteq 1} \frac{1-x^3}{1-x} \equiv \lim_{x \doteq 1} (1 + x + x^2) = 3.$$

EXERCISES

Evaluate each of the following indeterminate forms.

1. $\dfrac{x^2-5x+6}{x^2-4x+3}$, when $x \doteq 3$.
2. $\dfrac{x^2-1}{x^2-3x+2}$, when $x \doteq 1$.
3. $\dfrac{x^n-1}{x-1}$, when $x \doteq 1$.
4. $\dfrac{(x+h)^n - x^n}{h}$, when $h \doteq 0$.

153. Other Indeterminate Forms. There are various other indeterminate forms which in many cases may be reduced to the standard form $0/0$.

Example 1. The expression

$$\frac{x}{x-1} - \frac{x^2}{x^2-1}$$

takes the indeterminate form $\infty - \infty$ when $x \doteq 1$. But when the indicated subtraction is performed, it takes the indeterminate form $0/0$, and is easily evaluated.

Example 2. The expression $(2x-3)/(3x-2)$ satisfies the identity

$$\frac{2x-3}{3x-2} \equiv \frac{\frac{1}{3x-2}}{\frac{1}{2x-3}}.$$

When x approaches infinity, the first expression takes the indeterminate form ∞/∞, while the second takes the indeterminate form $0/0$.

EXERCISES

1. Evaluate each of the following indeterminate forms.

(*a*) $\dfrac{4}{x^2-1}-\dfrac{2}{x-1}$, when $x \doteq 1$.

(*b*) $\dfrac{2x^2-5x+4}{3x^2-4x-3}$, when x increases without limit.

[HINT. Divide numerator and denominator by x^2.]

2. Prove that when x becomes infinite the quotient

$$\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{b_0x^n+b_1x^{n-1}+\cdots+b_n}$$

of two rational integral functions approaches the limit 0, or a_0/b_0, or becomes infinite, according as $m<n$, $m=n$, or $m>n$.

3. Prove that for sufficiently large negative values of x, the graph of $a_0x^n+a_1x^{n-1}+\cdots+a_n, (a_0>0)$, will be found above the x-axis when n is even and below the x-axis when n is odd, and for sufficiently large positive values, it will be found above the x-axis, n even *or* n odd.

154. The Derivative of a Power Function. Let $f(x)$ be any function of x. The limit

$$\lim_{h\doteq 0}\frac{f(x+h)-f(x)}{h}, \tag{11}$$

is called the ***derivative*** of $f(x)$ and is usually denoted by $f'(x)$. The derivative of a power function cx^n (§ 53) is easily found by the preceding methods. For the power function the derivative is

$$\lim_{h\doteq 0}\frac{c(x+h)^n-cx^n}{h}\equiv c\lim_{h\doteq 0}\frac{(x+h)^n-x^n}{h}.$$

The expression on the right is precisely the indeterminate form of Ex. 4, p. 228. To evaluate it when n is a positive integer, note that, by (6), § 17,

$$(x+h)^n-x^n\equiv[x+h-x][(x+h)^{n-1}+x(x+h)^{n-2}+\cdots+x^{n-1}].$$

The first factor on the right is h so that the derivative of cx^n reduces to

$$\lim_{h \doteq 0} \left[(x+h)^{n-1} + x(x+h)^{n-2} + \cdots + x^{n-1}\right].$$

In the limit each one of the n terms within the bracket reduces to x^{n-1} and consequently the derivative of cx^n is

$$ncx^{n-1}. \tag{12}$$

A slight modification of the reasoning leads to a result having exactly the same form when n is not a positive integer.

Hence the rule for finding the derivative of the power function is as follows. *Diminish the exponent of x by 1 and multiply the result by the original exponent.*

155. The Derivative of a Polynomial. Let us consider the polynomial

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n.$$

According to the definition, the derivative is

$$f'(x) = \lim_{h \doteq 0} \frac{a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_n - (a_0x^n + \cdots + a_n)}{h}.$$

The limit on the right reduces directly to the sum

$$\lim_{h \doteq 0} \frac{a_0(x+h)^n - a_0x^n}{h} + \lim_{h \doteq 0} \frac{a_1(x+h)^{n-1} - a_1x^{n-1}}{h} + \cdots$$
$$+ \lim_{h \doteq 0} \frac{a_{n-1}(x+h) - a_{n-1}x}{h}.$$

This sum is precisely the sum of the derivatives of the power functions

$$a_0x^n,\ a_1x^{n-1},\ a_2x^{n-2} \cdots a_{n-1}x.$$

The derivative of the polynomial is therefore

$$f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a^{n-1}. \tag{13}$$

EXERCISES

1. Find the derivative of the function

$$f(x) = 2x^4 - 5x^3 + 3x^2 + 4x + 6.$$

SOLUTION. By what precedes the derivative is the sum of the derivatives of the power functions $2x^4$, $-5x^3$, $3x^2$, $4x$. By § 154 this sum is

$$f'(x) = 8x^3 - 15x^2 + 6x + 4.$$

Find the derivatives of each of the following functions.

2. $6x^3 - 7x^2 + 14$.

3. $8x^4 - 16x^2 + 14x$.

4. $x^n - a^n$.

5. $x^n + a^n$.

156. Geometric Interpretation of the Derivative. Slope of a Curve. Let $f(x)$ be a polynomial, which, for definiteness, may be assumed to be of degree 3. Its graph will, therefore, be similar to the curve in Fig. 39.

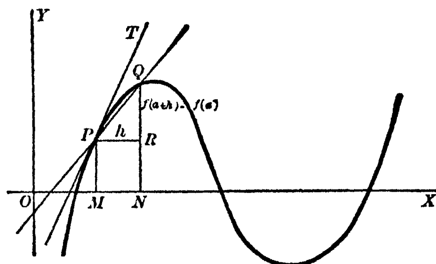


FIG. 39.

Let P and Q be two points on the curve whose abscissas are $OM = a$ and $ON = a + h$. The ordinates are $f(a)$ and $f(a+h)$ respectively. Draw the chord PQ and the line PR parallel to the x-axis meeting the ordinate NQ in R. Then RQ/PR is the slope of the chord (§ 49). But $RQ = NQ - NR = NQ - MP = f(a+h) - f(a)$ and $PR = MN = h$. Therefore the slope of the chord is

$$\frac{f(a+h) - f(a)}{h}. \tag{14}$$

If the point Q move along the curve toward P, the point R will approach P and h will approach zero. The limiting position of the chord will be the tangent line at P and the slope

of the chord becomes the slope of the tangent. The slope of the tangent is therefore given by the formula

(15) $$\textbf{Slope} = \lim_{h \doteq 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

The slope of a curve at a point is defined as the slope of the tangent at the point. *Therefore the geometric interpretation of the derivative of $f(x)$ at a point $x = a$ is that it is the slope of the curve $y = f(x)$ at the point whose abscissa is a.*

EXERCISES

1. Draw a line having the same slope as the curve whose equation is $y = x^2 - 5x + 6$ has at the point for which $x = 3$.

2. Find the points at which the tangent to the curve $y = x^3 - 7x^2 + 6 - 14$ is parallel to the x-axis.

157. Maxima and Minima. In § 52 the maximum value of a function was defined as a value larger than any other value determined by nearby values of the independent variable. In symbols, $f(a)$ is a maximum if for values of h sufficiently small, $f(a - h)$ and $f(a + h)$ are both less than $f(a)$.

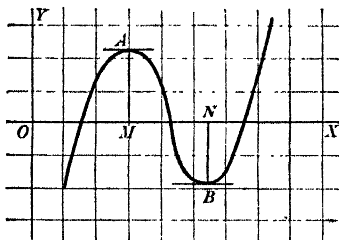


FIG. 40.

The definition of a minimum differs only in the substitution of the phrase *less than* for the phrase *larger than.*

In the adjoining figure (Fig. 40) the ordinate at the point A represents a maximum, and the ordinate at the point B a minimum, value of the function.

Clearly, at either a maximum or a minimum point the slope is zero. But the slope is given by $f'(x)$. Therefore the maximum and minimum values of $f(x)$ correspond to the values of x which satisfy the equation

$$f'(x) = 0. \tag{16}$$

The values of x corresponding to a maximum or a minimum may be found if the equation (16) can be solved.

Example. Let

$$f(x) \equiv x^3 - 9x^2 + 23x - 15.$$

Then

$$f(x) \equiv 3x^2 - 18x + 23.$$

The roots of $f'(x) = 0$ are found to be, approximately, $x = 1.85$ and $x = 4.15$. When these values are substituted for x in $f(x)$, the maximum and minimum values of the function are found to be approximately

$$f(1.85) = 3.1 \text{ and } f(4.15) \equiv -3.1.$$

Brief consideration of the function shows that the first is a maximum and the second is a minimum, for the graph crosses the y-axis at $y = -15$.

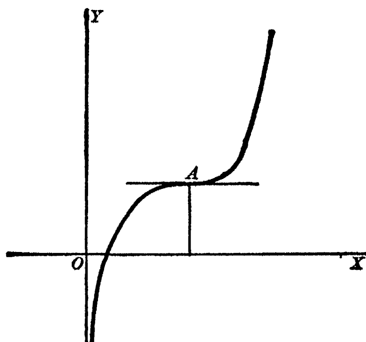


Fig. 41.

The condition $f'(x) = 0$ is necessary, but not sufficient, to insure a maximum or a minimum. A glance at Fig. 41 will show that a curve may have a point like the point A for which the slope is zero but the point is, nevertheless, neither a maximum nor a minimum.

EXERCISES

1. Find the maximum and minimum values of each of the following functions.

(*a*) $x^3 - 6x^2 + 9x - 1.$

(*b*) $x^4 - 15.$

(*c*) $2x^3 - 15x^2 + 36x - 14.$

(*d*) $x^3 - 15.$

[HINT. The function $x^3 - 15$ has neither a maximum nor a minimum. Verify the statement.]

2. What is the capacity of the largest box that can be made from a piece of pasteboard 24 by 36 inches by cutting a square from each corner?

3. Show that if an equation $f(x) = 0$ has a real double root it is geometrically evident that this root satisfies the two equations $f(x) = 0$ and $f'(x) = 0$.

4. Find the rectangle of maximum area which may be inscribed in a semicircle with radius r.

CHAPTER XVIII

INFINITE SERIES

158. Series with Constant Terms. If

(1) $$u_1, \quad u_2, \quad u_3, \quad \cdots, \quad u_n, \quad \cdots$$

be an infinite sequence, the expression

(2) $$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is called an ***infinite series.*** The term u_n is called the ***general term,*** or the ***type term.*** From the type term alone the series may be constructed. For example, if in the geometric progression

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n},$$

the number n be allowed to increase without bound, we obtain the infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots,$$

which has for its terms the terms of the infinite sequence

$$\frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \cdots, \quad \frac{1}{2^n}, \quad \cdots.$$

The terms correspond respectively to the values, $n = 1, 2, 3, 4, \cdots$.

Again, when the binomial expansion for $(1.01)^{1/4}$ is found, the result is the infinite series

$$1 + .0025 - .000009375 + .0000000546875 -, \cdots.$$

For this series the general term is found by finding the nth term of the binomial expansion.

In the case of the geometric series, it is possible to find the expression for the sum of n terms and by means of this sum to find the limit (if it exists) of the sum as n increases. In the general case it is not possible to find the sum of n terms, much less the limit of the sum. We are concerned mainly with the question of the existence of a limit for the sum of n terms as n increases.

Let

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n, \tag{3}$$

or more briefly,

$$s_n = \sum_1^n{}_i\, u_i, \tag{4}$$

where the sign Σ denotes the sum of such terms as the sample term that follows it. When the terms of the series are constant, s_n is a variable which depends for its value upon n alone. If, now, n be increased indefinitely, one of the three following cases will necessarily occur.

(1) The sum s_n may approach a finite number as a limit, as in the case of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots.$$

(2) The sum may increase beyond all bounds, as in the case of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots.$$

(3) The sum may take one of several values depending upon the form of n, as in the series

$$1+1-1-1+1+1-1-1+\cdots,$$

whose sum is 1, 2, or 0, according as n has the form $4r \pm 1$, $4r+2$, or $4r$.

In the first case the series is said to be ***convergent***, in the second, ***divergent***; in the third, ***oscillating***. It is simpler to put divergent and oscillating series into a single class and call both of them *divergent*.

A series is convergent if a finite limit exists for s_n as n increases; otherwise, it is divergent.

If $\lim_{n=\infty} s_n = S$, S is called the sum* of the series, and it is customary to write

(5) $$S = u_1 + u_2 + u_3 + \cdots.$$

159. Criterion for Convergence. The sums

(6) $$s_1, \quad s_2, \quad s_3, \quad \cdots, \quad s_n, \quad \cdots$$

of 1, 2, 3, ⋯ n ⋯ terms, form a sequence. The criterion for convergence of a series with real terms is therefore identical with the criterion for convergence of a sequence. When the *General Principle of Convergence* (§ 148) is applied to the sequence of sums

$$s_1, \quad s_2, \quad s_3, \quad s_4, \quad \cdots, \quad s_n, \quad \cdots$$

it takes the following form.

The necessary and sufficient condition for the convergence of a series is, that corresponding to any given positive number ϵ, a value for n may be found, such that all of the absolute values,

(7) $$|s_{n+1} - s_n|, \quad |s_{n+2} - s_n|, \quad |s_{n+3} - s_n|, \quad \cdots \quad |s_{n+p} - s_n|, \quad \cdots,$$

are less than ϵ.

This criterion may be expressed more briefly by writing

(8) $$|s_{n+p} - s_n| < \epsilon,$$

for every positive integral value of p.

A still different form is

(9) $$|u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}| < \epsilon,$$

for every positive integral value of p.

* The student should note that this use of the word "sum" is wholly different from that implied in the early chapters of this book. Here "sum" means the limit of a sequence of sums.

The form (8) is equivalent to the equation

$$(10) \qquad \lim_{n=\infty} |u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}| = 0,$$

for every value of p. The condition given in any one of the forms (7), (8), (9), or (10) is known as *Cauchy's criterion for convergence.*

From Cauchy's criterion it is at once apparent that the terms of a convergent series must approach zero as n increases. For, when $p = 1$, equation (10) becomes

$$\lim_{n=\infty} u_{n+1} = 0.$$

This condition is not sufficient as a single example will show.

Thus the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

has the general term $1/n$. Consequently,

$$\lim_{n=\infty} u_{n+1} = \lim_{n=\infty} \frac{1}{n+1} = 0.$$

But the series is not convergent. For, the sums obtained by taking the third and fourth terms, the fifth to the eighth term, inclusive, the ninth to the sixteenth, and so on, are all greater than 1/2. In this way it is possible to group the terms in such manner that an unlimited number of groups of terms are obtained, each group having the sum of its terms greater than 1/2. Hence, the series is divergent.

Only the most limited use can be made of divergent series. Consequently discussions relating to series have to do principally with convergent series, and with tests for convergence. Two of the most important tests are given below.

160. The Comparison Test. *If from and after a given term of a series with constant terms, every term bears a finite ratio to the corresponding terms of a comparison series known to be convergent, the series is convergent; if every term bears a finite ratio to the terms of a comparison series known to be divergent, the series is divergent.*

PROOF FOR CONVERGENT SERIES. Let

$$u_1 + u_2 + u_3 + \cdots$$

be a series to be tested, and

$$t_1 + t_2 + t_3 + \cdots$$

a series known to be convergent. Let U_n and T_n, respectively, be the sums of n terms of the two series. By hypothesis

$$u_1 = k_1 t_1, \quad u_2 = k_2 t_2, \quad u_3 = k_3 t_3, \cdots.$$

Let G be a finite number greater than the greatest k. Then

$$u_1 < Gt_1, \quad u_2 < Gt_2, \quad u_3 < Gt_3, \cdots.$$

Consequently,

$$\begin{aligned} s_{n+p} - s_n &= u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p} \\ &< G(t_{n+1} + t_{n+2} + t_{n+3} + \cdots + t_{n+p}). \end{aligned}$$

But by hypothesis n can be found such that the absolute value

$$|t_{n+1} + t_{n+2} + t_{n+3} + \cdots + t_{n+p}|$$

will be less than any assigned positive number, δ, then less than δ/G.

For such a value of n,

$$|s_{n+p} - s_n| < \delta.$$

Hence the series is convergent.

PROOF FOR DIVERGENT SERIES. Suppose the series

$$t_1 + t_2 + t_3 + \cdots$$

is divergent, and let g be a positive number smaller than the smallest k. Then

$$u_1 > gt_1, \quad u_2 > gt_2, \cdots,$$

and

$$U_n > g(t_1 + t_2 + \cdots + t_n);$$

or,

$$U_n > gT_n.$$

But T_n increases without limit. Therefore U_n increases without limit and the series

$$u_1 + u_2 + u_3 + \cdots$$

is divergent.

COROLLARY. *If*

$$\Sigma t_i = t_1 + t_2 + t_3 + \cdots$$

be a convergent series with positive terms, and

$$\Sigma u_i = u_1 + u_2 + u_3 + \cdots$$

be a series such that $u_i < t_i$, *for every value of* i, *then* Σu_i *is convergent; if, on the other hand, the series* Σt_i *is divergent and* $u_i > t_i$, *then the series* Σu_i *is divergent.*

The corollary is merely a restatement of the theorem for the cases, first where every k is less than 1, and second where every k is greater than 1.

161. Some Comparison Series. For the comparison test it is necessary to have at hand a number of series whose character is known. The following are some of the commoner series used for this purpose:

1. *Any geometric series*

$$ar + ar^2 + ar^3 + \cdots \tag{11}$$

is convergent when r *is less than* 1, *and divergent when* $r \geqq 1$.

For, the sum of n terms is, by (6), § 128,

$$S_n = \frac{ar^n - a}{r - 1}. \tag{12}$$

This sum has a limit $a/(1 - r)$ for $r < 1$, but no limit for $r \geqq 1$.

2. *The series*

$$\frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots + \frac{1}{n^k} + \cdots \tag{13}$$

is convergent for $k > 1$ *and divergent for* $k \leq 1$.

The terms may be grouped as follows:

$$(14)\qquad \frac{1}{1^k}+\frac{1}{2^k}+\left[\frac{1}{3^k}+\frac{1}{4^k}\right]+\left[\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}+\frac{1}{8^k}\right]+\cdots$$
$$+\left[\frac{1}{(2^n+1)^k}+\frac{1}{(2^n+2)^k}+\cdots+\frac{1}{(2^{n+1})^k}\right]+\cdots.$$

Every one of the 2^n terms which go to make up the general terms of the series with the new grouping of the terms is less than $1/2^{nk}$. Consequently, the general term of the regrouped series is less than $2^n(1/2^{nk})$, that is, less than $1/2^{n(k-1)}$. The terms of the series in the new form are, after the second term, less than the corresponding terms of the series

$$(15)\qquad \frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{2^{k-1}}+\frac{1}{2^{2(k-1)}}+\cdots+\frac{1}{2^{n(k-1)}}+\cdots.$$

After the second term the series (15) is a geometric series with ratio $1/2^{k-1}$. This ratio is less than 1 for $k>1$. Therefore the series (14), or what is the same thing, the series (13), is convergent for $k>1$.

On the other hand, each of the 2^n terms which make up the general term of the regrouped series is equal to or greater than

$$\frac{1}{2^{(n-1)k}}$$

and the general term is therefore greater than

$$2^n\cdot\frac{1}{2^{(n+1)k}},\quad\text{or}\quad \frac{1}{2^k}\cdot\frac{1}{2^{n(k-1)}}.$$

Each term of the regrouped series is, after the second term, greater than the terms of the series

$$(16)\quad \frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{2^k}\cdot\frac{1}{2^{k-1}}+\frac{1}{2^k}\cdot\frac{1}{2^{2(k-1)}}+\cdots+\frac{1}{2^k}\cdot\frac{1}{2^{n(k-1)}}+\cdots.$$

After the second term the series (16) is a geometric series with ratio $1/2^{k-1}$. The ratio is equal to or greater than 1 when $k\leqq 1$. The series (13) is therefore divergent when $k\leqq 1$.

EXERCISES

1. Determine the character of the series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\cdots\frac{1}{n!}+\cdots.$$

SOLUTION. If the first two terms of the series be set aside, the terms of the new series will all be less than the corresponding terms of the convergent geometric series

$$1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\cdots$$

because the general term $1/n!$ is less than the general term $1/2^n$. The series is therefore convergent.

2. Determine the character of the series

$$\frac{1}{1}+\frac{2}{2}+\frac{2^2}{3}+\frac{2^3}{4}+\cdots.$$

SOLUTION. The general term is $2^{n-1}/n$ which is greater than the general term of the divergent series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots.$$

The series is therefore divergent.

3. Determine the character of the following series.

(a) $\frac{3}{1\cdot2\cdot2}+\frac{4}{2\cdot3\cdot2^2}+\frac{5}{3\cdot4\cdot2^3}+\frac{6}{4\cdot5\cdot2^4}+\cdots.$

(b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots.$

(c) $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\cdots.$

162. D'Alembert's Test-ratio Test. The ***test-ratio*** of a series $u_1+u_2+u_3+\cdots$ is the ratio of the general term to the term preceding it. It may be written u_n/u_{n-1}, or what amounts to the same thing, u_{n+1}/u_n.

THEOREM. *If in a series*

$$\Sigma u_i = u_1+u_2+u_3+\cdots,$$

with real positive terms, $\lim\limits_{n=\infty}(u_{n+1}/u_n)=t$ *is less than* 1, *the series is convergent; if* $t>1$, *the series is divergent. If* $t=1$, *the character of the series is not determined by this test.*

PROOF FOR CONVERGENT SERIES.

Let k be a number lying between t and 1 (Fig. 42). If $t = \lim_{n=\infty} (u_{n+1}/u_n)$ is less than 1, there will be a value m of n such that u_{m+1}/u_m and all subsequent ratios will be less than k, that is



FIG. 42.

$$(17) \qquad u_{m+1} < ku_m, \quad u_{m+2} < ku_{m+1}, \quad u_{m+3} < ku_{m+2}, \cdots.$$

When $u_{m+1}, u_{m+2}, u_{m+3}, \cdots$ are eliminated from the second, third, fourth, $\cdots$, inequalities, the set (17) may be written in the form

$$(18) \qquad u_{m+1} < ku_m, \quad u_{m+2} < k^2u_m, \quad u_{m+3} < k^3u_m, \cdots.$$

The inequalities (18) state that beginning with the $(m+1)$st term the terms of the series Σu_i are less than the corresponding terms of the geometric series

$$(19) \qquad ku_m + k^2u_m + k^3u_m + \cdots.$$

When $k < 1$ the series (19) is convergent, and consequently the series Σu_i is convergent.

PROOF FOR DIVERGENT SERIES. By a process of reasoning similar to that used in the first part of the proof it may be shown that when $t > k > 1$,

$$u_{m+1} > ku_m, \quad u_{m+2} > ku_{m+1}, \quad u_{m+3} > ku_{m+2}, \cdots;$$

and further, that

$$u_{m+1} > ku_m, \quad u_{m+2} > k^2u_m, \quad u_{m+3} > k^3u_m, \cdots.$$

It follows that from the mth term onward, the terms of the series Σu_i are greater than the terms of the *divergent* geometric series

$$ku_m + k^2u_m + k^3u_m + \cdots.$$

Hence the series is divergent.

When $t = 1$ the test fails, since $t = 1$ for some convergent and for some divergent series. For example, the series

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots,$$

for which $t=1$, is convergent, while the series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots,$$

for which $t=1$, is divergent. (See § 161.)

EXERCISES

Determine the character of the following series.

1. $\frac{1}{1\cdot 2}\cdot\frac{1}{2}+\frac{1}{2\cdot 3}\cdot\frac{1}{2^2}+\frac{1}{3\cdot 4}\cdot\frac{1}{2^3}+\cdots.$

2. $\frac{1}{1\cdot 2}\cdot\frac{3}{2}+\frac{1}{2\cdot 3}\cdot\frac{3^2}{2^2}+\frac{1}{3\cdot 4}\cdot\frac{3^3}{2^3}+\cdots.$

3. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots.$

4. $\frac{1}{1\cdot 3}\cdot\frac{1}{2}+\frac{1}{3\cdot 5}\frac{1}{2^2}+\frac{1}{5\cdot 7}\frac{1}{2^3}+\cdots.$

5. $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^2}+\frac{4}{3^2}+\cdots.$

6. $\frac{1\cdot 2}{1000^2}+\frac{1\cdot 2\cdot 3}{1000^3}+\frac{1\cdot 2\cdot 3\cdot 4}{1000^4}+\cdots.$

163. Series with Negative, or with Imaginary Terms. A series

$$u_1+u_2+u_3+\cdots$$

composed of positive and negative terms, or of complex terms, is said to be ***absolutely convergent***, if the series

$$(20)\qquad |u_1|+|u_2|+|u_3|+\cdots,$$

formed of the absolute values of the terms of the original series, is convergent. If the original series is convergent but the series of absolute terms is divergent, the original series is said to be ***semi-convergent***.

EXAMPLE 1. The series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}\cdots$$

is absolutely convergent since the series of absolute values

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

is convergent.

EXAMPLE 2. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

is semi-convergent since the series of absolute values

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

is divergent.

EXAMPLE 3. The series

$$1+\frac{1}{3+4i}+\frac{1}{(3+4i)^2}+\frac{1}{(3+4i)^3}+\cdots$$

is absolutely convergent since the series of absolute values is the convergent geometric series

$$1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots.$$

164. Power Series. A series whose terms are power functions with increasing exponents is called a **power series.** Such a series has the form

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots. \tag{21}$$

A power series may be convergent for some values of x and divergent for others.

The test-ratio test is the most useful means for determining the values of x for which the series is convergent or divergent. For example, one of the simplest power series is the series

$$1+x+2x^2+3x^3+\cdots.$$

The general term is nx^n and the test-ratio is $(n+1)x^{n+1}/(nx^n)$, which reduces to $(n+1)x/n$. The limit of the test-ratio is therefore x. Consequently, the series is convergent for

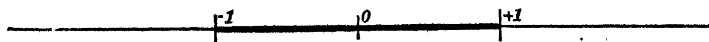


FIG. 43.

$|x|<1$. The inequality $|x|<1$ is satisfied for every value of x which lies between $+1$ and -1, Fig. 43; or, as it is usually expressed, for values of x such that $-1<x<+1$.

The test-ratio for the general power series is $a_{n+1}x/a_n$. In order that the series may be convergent, it is necessary that

$$t = \lim_{n=\infty} \left| \frac{a_{n+1}x}{a_n} \right| < 1,$$

or that

$$|x| < \left| \lim_{n=\infty} \frac{a_n}{a_{n+1}} \right|.$$

The last inequality may be written in the form

$$(22) \qquad -\lim_{n=\infty} \left| \frac{a_n}{a_{n+1}} \right| < x < \lim_{n=\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

If $\lim_{n=\infty} (a_{n+1}/a_n) = 0$, the series is convergent for every finite value of x.

EXERCISES

Find the range of values of x for which the following power series are convergent.

1. $1 + 2x + 2^2x^2 + 2^3x^3 + \cdots$.

2. $mx + \frac{m(m-2)}{1 \cdot 2}x^2 + \frac{m(m-2)(m-4)}{1 \cdot 2 \cdot 3}x^3 + \cdots$.

3. $1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots$.

4. $1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \cdots$.

5. $x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \cdots$.

6. $1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^3 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots$.

7. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

165. The Binomial Series. The mth power of a binomial may be written in the form

(23) $$(a+x)^m = a^m\left(1+\frac{x}{a}\right)^m.$$

By § 118 the second factor may be expanded in the form

(24) $$\left(1+\frac{x}{a}\right)^m = 1 + m\frac{x}{a} + \frac{m(m-1)}{1\cdot 2}\left(\frac{x}{a}\right)^2 + \cdots + \frac{m(m-1)\cdots(m-\overline{n-2})}{1\cdot 2\cdot 3\cdots(n-1)}\left(\frac{x}{a}\right)^{n-1} + \cdots.$$

The limit of the ratio of the $(n+1)$st term to the nth is

(25) $$\lim_{n=\infty}\frac{\dfrac{m(m-1)\cdots(m-\overline{n-1})}{1\cdot 2\cdot 3\cdots n}\left(\dfrac{x}{a}\right)^n}{\dfrac{m(m-1)\cdots(m-\overline{n-2})}{1\cdot 2\cdot 3\cdots(n-1)}\left(\dfrac{x}{a}\right)^{n-1}} = \lim_{n=\infty}\frac{m-n-1}{n}\frac{x}{a} = -\frac{x}{a}.$$

In order that the series may be convergent, the absolute value of this limit must be less than 1, that is

(26) $$\left|\frac{x}{a}\right| < 1, \quad \text{or} \quad -a < x < a,$$

if a is positive. It can be proved that the series (24) represents the expansion of $(1+x/a)^m$ when, and only when, the condition (26) is satisfied.

166. The Exponential Series. In § 136 the expression $\lim_{n=\infty}(1+x/n)^n$ occurred, and it was there denoted by e^x. It can be proven rigorously that

(27) $$e^x = \lim_{n=\infty}\left(1+\frac{x}{n}\right)^n = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots + \frac{x^n}{1\cdot 2\cdot 3\cdots n} + \cdots.$$

It is easily shown that this power series is convergent for *all* values of x. (See Ex. 3, § 164.)

167. The Logarithmic Series. The computation of logarithms depends directly, or indirectly, upon the series

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots. \tag{28}$$

This power series converges (Ex. 7, § 164) for values of x such that

$$-1<x<+1.$$

The series (28) converges slowly, that is, it requires many terms to give a good approximation to the value of $\log(1+x)$. A series which, for small values of x, converges more rapidly may be obtained by subtracting from (28) the series

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots, \tag{29}$$

which is obtained from (28) by changing the sign of x. The new series thus obtained is

$$\log_e\frac{1+x}{1-x}=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]. \tag{30}$$

If, for example, $x=1/10$, three terms of the series (30) will give the Napierian logarithm of $11/9$ to the sixth decimal place. We find

$$\log_e 11/9=0.200671.$$

To find the common logarithm of 11/9, the result just obtained must be multiplied by the ***modulus*** for reducing Napierian to common logarithms. This modulus is, to the sixth decimal place,

$$M=0.434294. \tag{31}$$

It follows that $\log_{10}(11/9)=0.087149.$

168. Series as a Means of Computation. One of the most important uses to which series are put is that of finding the approximate numerical values of functions whose values otherwise can be found only with great difficulty, or perhaps not at

all. A few examples will make the matter clear. The use of series in the computation of logarithms was shown in § 167.

Among the most important functions in elementary mathematics are the ***sine*** and ***cosine*** of an angle. If the angle x be measured in radians, the sine of x is given by the power series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{32}$$

The sine of one radian, which is approximately 57°.29578, is given by the series

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots.$$

Four terms give

$$\sin 1 = .8415,$$

a result that is correct as far as it goes, since further terms would not affect the fourth decimal.

To find sin .1 to the fourth decimal place only two terms of the series would be required.

Another series of very great importance in both pure and applied mathematics is the power series which represents the ***exponential function.*** This series is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Consequently

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots. \tag{33}$$

Ten terms of the series (33) give to the nearest ten-millionth,

$$e = 2.7182818.$$

This result is correct to the seventh decimal place.

Roots of e, such as $\sqrt{e} = e^{.5}$ and $\sqrt[10]{e} = e^{.1}$, may be computed even more rapidly.

EXERCISES

1. Find approximate numerical values for each of the following functions given by power series.

(*a*) $\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \cdots$ when $x = 1$ radian or $57^\circ.29578$.

(*b*) $\sqrt{e} = e^{.5} = 1 + \frac{.5}{1} + \frac{(.5)^2}{1 \cdot 2} + \frac{(.5)^3}{1 \cdot 2 \cdot 3} + \cdots$.

(*c*) $\sqrt[10]{e} = e^{.1}$.

2. Compute the value of $(1 + .06)^{1/12}$ to the fourth decimal place.

3. Extract the seventh root of 1.02 by the binomial series.

4. Extract the seventh root of .99 by the binomial series.

5. By means of the series (30) of § 167, and the modulus, $M = 0.434294$, find the Napierian and the common logarithms of $\frac{5}{4}$.

6. Find the Napierian and the common logarithms of 3.

7. In trigonometry it is shown that

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8},$$

and that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

From these data find the value of π to four decimal places.

Certain Convenient Values for $n = 1$ to $n = 10$

n	$1/n$	$\sqrt{n}$	$\sqrt[3]{n}$	$n!$	$1/n!$	$\log_{10} n$
1	1.000000	1.00000	1.00000	1	1.0000000	0.000000000
2	0.500000	1.41421	1.25992	2	0.5000000	0.301029996
3	0.333333	1.73205	1.44225	6	0.1666667	0.477121255
4	0.250000	2.00000	1.58740	24	0.0416667	0.602059991
5	0.200000	2.23607	1.70998	120	0.0083333	0.698970004
6	0.166667	2.44949	1.81712	720	0.0013889	0.778151250
7	0.142857	2.64575	1.91293	5040	0.0001984	0.845098040
8	0.125000	2.82843	2.00000	40320	0.0000248	0.903089987
9	0.111111	3.00000	2.08008	362880	0.0000028	0.954242509
10	0.100000	3.16228	2.15443	3628800	0.0000003	1.000000000

Logarithms of Important Constants

n = number	Value of n	$\log_{10} n$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Napierian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812262
$\pi \div 180$ = radians in 1°	0.01745329	8.24187738
$\pi \div 10800$ = radians in 1′	0.0002908882	6.46372613
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811152	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37 (exact legal value)	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g (average value)	32.16 ft./sec./sec.	1.5073
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953586
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.907
cu. in. in 1 (U. S.) gallon	231 (exact legal value)	2.3636120
ft. lb. per sec. in 1 H. P.	550. (exact legal value)	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1.8810445
watts in 1 H. P.	745.957	2.8727135

Table D — Compound Interest Table

AMOUNT OF ONE DOLLAR PRINCIPAL WITH COMPOUND INTEREST AT VARIOUS RATE

YEARS	2½%	3%	3½%	4%	4½%	5%	5½%	6%	6½%	7%	8%
1	$1.025	$1.030	$1.035	$1.040	$1.045	$1.050	$1.055	$1.060	$1.065	$1.070	$1.080
2	1.051	1.061	1.071	1.082	1.092	1.103	1.113	1.124	1.134	1.145	1.166
3	1.077	1.093	1.109	1.125	1.141	1.158	1.174	1.191	1.208	1.225	1.260
4	1.104	1.126	1.148	1.170	1.193	1.216	1.239	1.262	1.286	1.311	1.360
5	1.131	1.159	1.188	1.217	1.246	1.276	1.307	1.338	1.370	1.403	1.469
6	1.160	1.194	1.229	1.265	1.302	1.340	1.379	1.419	1.459	1.501	1.587
7	1.189	1.230	1.272	1.316	1.361	1.407	1.455	1.504	1.554	1.606	1.714
8	1.218	1.267	1.317	1.369	1.422	1.477	1.535	1.594	1.655	1.718	1.851
9	1.249	1.305	1.363	1.423	1.486	1.551	1.619	1.689	1.763	1.838	1.999
10	1.280	1.344	1.411	1.480	1.553	1.629	1.708	1.791	1.877	1.967	2.159
11	1.312	1.384	1.460	1.539	1.623	1.710	1.802	1.898	1.999	2.105	2.332
12	1.345	1.426	1.511	1.601	1.696	1.796	1.901	2.012	2.129	2.252	2.518
13	1.379	1.469	1.564	1.665	1.772	1.886	2.006	2.133	2.267	2.410	2.720
14	1.413	1.513	1.619	1.732	1.852	1.980	2.116	2.261	2.415	2.579	2.937
15	1.448	1.558	1.675	1.801	1.935	2.079	2.232	2.397	2.572	2.759	3.172
16	1.485	1.605	1.734	1.873	2.022	2.183	2.355	2.540	2.739	2.952	3.426
17	1.522	1.653	1.795	1.948	2.113	2.292	2.485	2.693	2.917	3.159	3.700
18	1.560	1.702	1.857	2.026	2.208	2.407	2.621	2.854	3.107	3.380	3.996
19	1.599	1.754	1.923	2.107	2.308	2.527	2.766	3.026	3.309	3.617	4.316
20	1.639	1.806	1.990	2.191	2.412	2.653	2.918	3.207	3.524	3.870	4.661
21	1.680	1.860	2.059	2.279	2.520	2.786	3.078	3.400	3.753	4.141	5.034
22	1.722	1.916	2.132	2.370	2.634	2.925	3.248	3.604	3.997	4.430	5.437
23	1.765	1.974	2.206	2.465	2.752	3.072	3.426	3.820	4.256	4.741	5.871
24	1.809	2.033	2.283	2.563	2.876	3.225	3.615	4.049	4.533	5.072	6.341
25	1.854	2.094	2.363	2.666	3.005	3.386	3.813	4.292	4.828	5.427	6.848
26	1.900	2.157	2.446	2.772	3.141	3.556	4.023	4.549	5.142	5.807	7.396
27	1.948	2.221	2.532	2.883	3.282	3.733	4.244	4.822	5.476	6.214	7.988
28	1.996	2.288	2.620	2.999	3.430	3.920	4.478	5.112	5.832	6.649	8.627
29	2.046	2.357	2.712	3.119	3.584	4.116	4.724	5.418	6.211	7.114	9.317
30	2.098	2.427	2.807	3.243	3.745	4.322	4.984	5.743	6.614	7.612	10.063
31	2.150	2.500	2.905	3.373	3.914	4.538	5.258	6.088	7.044	8.145	10.868
32	2.204	2.575	3.007	3.508	4.090	4.765	5.547	6.453	7.502	8.715	11.737
33	2.259	2.652	3.112	3.648	4.274	5.003	5.852	6.841	7.990	9.325	12.676
34	2.315	2.732	3.221	3.794	4.466	5.253	6.174	7.251	8.509	9.978	13.690
35	2.373	2.814	3.334	3.946	4.667	5.516	6.514	7.686	9.062	10.677	14.785
36	2.433	2.898	3.450	4.104	4.877	5.792	6.872	8.147	9.651	11.424	15.968
37	2.493	2.985	3.571	4.268	5.097	6.081	7.250	8.636	10.279	12.224	17.246
38	2.556	3.075	3.696	4.439	5.326	6.385	7.649	9.154	10.947	13.079	18.625
39	2.620	3.167	3.825	4.616	5.566	6.705	8.069	9.704	11.658	13.995	20.115
40	2.685	3.262	3.959	4.801	5.816	7.040	8.513	10.286	12.416	14.974	21.725
41	2.752	3.360	4.098	4.993	6.078	7.392	8.982	10.903	13.223	16.023	23.462
42	2.821	3.461	4.241	5.193	6.352	7.762	9.476	11.557	14.083	17.144	25.339
43	2.892	3.565	4.390	5.400	6.637	8.150	9.997	12.250	14.998	18.344	27.367
44	2.964	3.671	4.543	5.617	6.936	8.557	10.546	12.985	15.973	19.628	29.556
45	3.038	3.782	4.702	5.841	7.248	8.985	11.127	13.765	17.011	21.002	31.920
46	3.114	3.895	4.867	6.075	7.574	9.434	11.739	14.590	18.117	22.473	34.474
47	3.192	4.012	5.037	6.318	7.915	9.906	12.384	15.466	19.294	24.046	37.232
48	3.271	4.132	5.214	6.571	8.271	10.401	13.065	16.394	20.549	25.729	40.211
49	3.353	4.256	5.396	6.833	8.644	10.921	13.784	17.378	21.884	27.530	43.427
50	3.437	4.384	5.585	7.107	9.033	11.467	14.542	18.420	23.307	29.457	46.902

PRESENT VALUE OF ONE DOLLAR DUE IN A CERTAIN NUMBER OF YEARS

YEARS	2½%	3%	3½%	4%	4½%	5%	5½%	6%	6½%	7%	8%
1	$.9756	$.9709	$.9662	$.9615	$.9569	$.9524	$.9479	$.9434	$.9390	$.9346	$.9259
2	.9518	.9426	.9335	.9246	.9157	.9070	.8985	.8900	.8817	.8734	.8573
3	.9286	.9151	.9019	.8890	.8763	.8638	.8516	.8396	.8278	.8163	.7938
4	.9060	.8885	.8714	.8548	.8386	.8227	.8072	.7921	.7773	.7629	.7350
5	.8839	.8626	.8420	.8219	.8025	.7835	.7651	.7473	.7299	.7130	.6806
6	.8623	.8375	.8135	.7903	.7679	.7462	.7253	.7050	.6853	.6663	.6302
7	.8413	.8131	.7860	.7599	.7348	.7107	.6874	.6651	.6435	.6227	.5835
8	.8207	.7894	.7594	.7307	.7032	.6768	.6516	.6274	.6042	.5820	.5403
9	.8007	.7664	.7337	.7026	.6729	.6446	.6176	.5919	.5674	.5439	.5002
10	.7812	.7441	.7089	.6756	.6439	.6139	.5854	.5584	.5327	.5083	.4632
11	.7621	.7224	.6849	.6496	.6162	.5847	.5549	.5268	.5002	.4751	.4289
12	.7436	.7014	.6618	.6246	.5897	.5568	.5260	.4970	.4697	.4440	.3971
13	.7254	.6810	.6394	.6006	.5643	.5303	.4986	.4688	.4410	.4150	.3677
14	.7077	.6611	.6178	.5775	.5400	.5051	.4726	.4423	.4141	.3878	.3405
15	.6905	.6419	.5969	.5553	.5167	.4810	.4479	.4173	.3888	.3624	.3152
16	.6736	.6232	.5767	.5339	.4945	.4581	.4246	.3936	.3651	.3387	.2919
17	.6572	.6050	.5572	.5134	.4732	.4363	.4025	.3714	.3428	.3166	.2703
18	.6412	.5874	.5384	.4936	.4528	.4155	.3815	.3503	.3219	.2959	.2502
19	.6255	.5703	.5202	.4746	.4333	.3957	.3616	.3305	.3022	.2765	.2317
20	.6103	.5537	.5026	.4564	.4146	.3769	.3427	.3118	.2838	.2584	.2145
21	.5954	.5375	.4856	.4388	.3968	.3589	.3249	.2942	.2665	.2415	.1987
22	.5809	.5219	.4692	.4220	.3797	.3418	.3079	.2775	.2502	.2257	.1839
23	.5667	.5067	.4533	.4057	.3634	.3256	.2919	.2618	.2349	.2109	.1703
24	.5529	.4919	.4380	.3901	.3477	.3101	.2767	.2470	.2206	.1971	.1577
25	.5394	.4776	.4231	.3751	.3327	.2953	.2622	.2330	.2071	.1843	.1460
26	.5262	.4637	.4088	.3607	.3184	.2812	.2486	.2198	.1945	.1722	.1352
27	.5134	.4502	.3950	.3468	.3047	.2678	.2356	.2074	.1826	.1609	.1252
28	.5009	.4371	.3817	.3335	.2916	.2551	.2233	.1956	.1715	.1504	.1159
29	.4887	.4243	.3687	.3207	.2790	.2429	.2117	.1846	.1610	.1406	.1073
30	.4767	.4120	.3563	.3083	.2670	.2314	.2006	.1741	.1512	.1314	.0994
31	.4651	.4000	.3442	.2965	.2555	.2204	.1902	.1643	.1420	.1228	.0920
32	.4538	.3883	.3326	.2851	.2445	.2099	.1803	.1550	.1333	.1147	.0852
33	.4427	.3770	.3213	.2741	.2340	.1999	.1709	.1462	.1252	.1072	.0789
34	.4319	.3660	.3105	.2636	.2239	.1904	.1620	.1379	.1175	.1002	.0730
35	.4214	.3554	.3000	.2534	.2143	.1813	.1535	.1301	.1103	.0937	.0676
36	.4111	.3450	.2898	.2437	.2050	.1727	.1455	.1227	.1036	.0875	.0626
37	.4011	.3350	.2800	.2343	.1962	.1644	.1379	.1158	.0973	.0818	.0580
38	.3913	.3252	.2706	.2253	.1878	.1566	.1307	.1092	.0914	.0765	.0537
39	.3817	.3158	.2614	.2166	.1797	.1491	.1239	.1031	.0858	.0715	.0497
40	.3724	.3066	.2526	.2083	.1719	.1420	.1175	.0972	.0805	.0668	.0460
41	.3633	.2976	.2440	.2003	.1645	.1353	.1113	.0917	.0756	.0624	.0426
42	.3545	.2890	.2358	.1926	.1574	.1288	.1055	.0865	.0710	.0583	.0395
43	.3458	.2805	.2278	.1852	.1507	.1227	.1000	.0816	.0667	.0545	.0365
44	.3374	.2724	.2201	.1781	.1442	.1169	.0948	.0770	.0626	.0509	.0338
45	.3292	.2644	.2127	.1712	.1380	.1113	.0899	.0727	.0588	.0476	.0313
46	.3211	.2567	.2055	.1646	.1320	.1060	.0852	.0685	.0552	.0445	.0290
47	.3133	.2493	.1985	.1583	.1263	.1010	.0808	.0647	.0518	.0416	.0209
48	.3057	.2420	.1918	.1522	.1209	.0961	.0765	.0610	.0487	.0389	.0249
49	.2982	.2350	.1853	.1463	.1157	.0916	.0726	.0576	.0457	.0363	.0230
50	.2909	.2281	.1791	.1407	.1107	.0872	.0688	.0543	.0429	.0339	.0213

Age	Number Living	Number Dying	Age	Number Living	Number Dying	Age	Number Living	Number Dying
x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
10	100 000	749	**40**	78 106	765	**70**	38 569	2391
11	99 251	746	**41**	77 341	774	**71**	36 178	2448
12	98 505	743	**42**	76 567	785	**72**	33 730	2487
13	97 762	740	**43**	75 782	797	**73**	31 243	2505
14	97 022	737	**44**	74 985	812	**74**	28 738	2501
15	96 285	735	**45**	74 173	828	**75**	26 237	2476
16	95 550	732	**46**	73 345	848	**76**	23 761	2431
17	94 818	729	**47**	72 497	870	**77**	21 330	2369
18	94 089	727	**48**	71 627	896	**78**	18 961	2291
19	93 362	725	**49**	70 731	927	**79**	16 670	2196
20	92 637	723	**50**	69 804	962	**80**	14 474	2091
21	91 914	722	**51**	68 842	1001	**81**	12 383	1964
22	91 192	721	**52**	67 841	1044	**82**	10 419	1816
23	90 471	720	**53**	66 797	1091	**83**	8 603	1648
24	89 751	719	**54**	65 706	1143	**84**	6 955	1470
25	89 032	718	**55**	64 563	1199	**85**	5 485	1292
26	88 314	718	**56**	63 364	1269	**86**	4 193	1114
27	87 596	718	**57**	62 104	1325	**87**	3 079	933
28	86 878	718	**58**	60 779	1394	**88**	2 146	744
29	86 160	719	**59**	59 385	1468	**89**	1 402	555
30	85 441	720	**60**	57 917	1546	**90**	847	385
31	84 721	721	**61**	56 371	1628	**91**	462	246
32	84 000	723	**62**	54 743	1713	**92**	216	137
33	83 277	726	**63**	53 030	1800	**93**	79	58
34	82 551	729	**64**	51 230	1889	**94**	21	18
35	81 822	732	**65**	49 341	1980	**95**	3	3
36	81 090	737	**66**	47 361	2070			
37	80 353	742	**67**	45 291	2158			
38	79 611	749	**68**	43 133	2243			
39	78 862	756	**69**	40 890	2321			

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